## 1. Warm-Up (9 points)

(a) Given that $\hat{A}$ and $\hat{B}$ are hermitian

(i) [1,5 points] Show that the expectation value $\langle[\hat{A}, \hat{B}]\rangle$ is imaginary.

(ii) [1,5 points] Are the following operators hermitian?
1.) $\hat{A}\hat{B}\hat{A}$ 2.) $e^{i\,\hat{A}}$ 3.) $e^{i\,[\hat{A},\,\hat{B}]}$

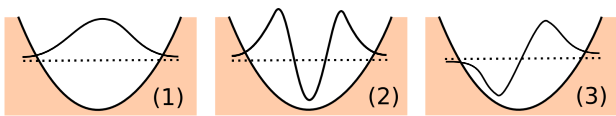
(b) [2 points] The parity operator is defined as $\hat{P}$: $\hat{P}\psi(\vec{x}) = \psi(-\vec{x})$. Show that $\hat{P}$ is hermitian.

[Hint: Use the definition of a hermitian conjugated operator.]

(c) [2 points] The state $|\psi\rangle$ of a one-dimensional system in coordinate representation is given by $\langle x|\psi\rangle = N \exp(-\kappa x^2)$ with $N \in \mathbb{R}$. Determine the normalization constant $N$, the expectation value $\langle\hat{X}\rangle$ and the standard deviation $\Delta X$.

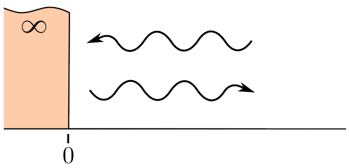
[Hint: $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\pi/\alpha}$.]

(d) [1 point] Sort the bound-state wave-functions in the figure by increasing energy. Reason your decision.




## 2. Total reflectance (6 points)

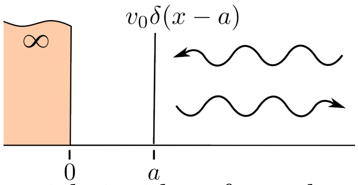
(a) [1 point] We consider the Schrödinger-equation for the one-dimensional potential




$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & x \geq 0 \end{cases}$$

Make an appropriate ansatz for the wave-function that fulfills the matching conditions at $x = 0$.

(b) A particle with energy $E = \frac{\hbar^2 k^2}{2m} > 0$ lives in the one-dimensional potential




$$V(x) = \begin{cases} \infty & x < 0 \\ v_0\delta(x-a) & x \geq 0 \end{cases}, \quad v_0,\, a > 0.$$

The particle incident from the right is reflected.

(i) [2 points] Make an appropriate ansatz for the wave-function $\psi(x)$ for the left-hand-side and the right-hand-side of the Delta-barrier.

(ii) [3 points] Formulate the matching conditions for the wave-functions at $x = 0$ and $x = a$. Exploit the fact that the particle is totally reflected, i.e. that the reflection-coefficient is given by $r = e^{i\varphi}$. What is the relation between phase $\varphi$ and energy $E$.

**See next page !!!**

**3. Particle in a potential well** (6 points)

Consider a particle in the one-dimensional potential

$$V(x) = \begin{cases} 0 & -\frac{a}{2} \le x \le \frac{a}{2} \\ \infty & \text{otherwise} \end{cases}, \qquad a > 0 \tag{1}$$

(a) [3 points] Derive the eigenenergies and the corresponding eigenstates $\phi_n(x)$ ($n \in \mathbb{N}$) of the Hamilton-Operator $\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}$.

(b) [3 points] Normalize the eigenstates and show that they are orthogonal.

<u>Hint</u>: The following relations could be useful
$\cos(x)\cos(y) = \frac{1}{2}\big[\cos(x-y) + \cos(x+y)\big]$, $\quad \sin(x)\sin(y) = \frac{1}{2}\big[\cos(x-y) - \cos(x+y)\big]$
$\sin(x)\cos(y) = \frac{1}{2}\big[\sin(x-y) + \sin(x+y)\big]$

**4. Two level system** (7 points)

The Hamiltonian represented in the basis $\{|u_1\rangle, |u_2\rangle\}$ is given by

$$\hat{H} = \begin{pmatrix} \varepsilon & -\Delta \\ -\Delta & \varepsilon \end{pmatrix} = \mathbb{1}\varepsilon - \hat{\sigma}_x \Delta \tag{2}$$

Here $\mathbb{1}$ is the $2 \times 2$ unit-matrix and $\hat{\sigma}_i$ are the Pauli-matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

(a) [1 point] Show that

(i) $\hat{\sigma}_y^2 = \mathbb{1}$, (ii) $\hat{\sigma}_x\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_x = i\hat{\sigma}_z$,

(b) [1 point] Show further that $e^{i\alpha\hat{\sigma}_y} = \cos(\alpha) + i\hat{\sigma}_y \sin(\alpha)$.

<u>Hint</u>: The following relations could be useful

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

(c) [3 points] Diagonalize the Hamiltonian and find a unitary matrix $\hat{U}$ so that $\hat{U}\hat{H}\hat{U}^\dagger = \text{diag}(E_1, E_2)$. Hint: $\hat{U}$ can be written as $\hat{U} = e^{i\alpha\hat{\sigma}_y}$. Determine $\alpha$, $E_1$ and $E_2$ and write the corresponding eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ as a linear combination of $|u_1\rangle$ and $|u_2\rangle$.

(d) [2 points] At the time-instant $t = 0$ the state of the system is initialized as $|\psi(t = 0)\rangle = |u_1\rangle$. Derive $|\psi(t)\rangle$ for $t > 0$. [ <u>Hint:</u> Express $|u_1\rangle$ by the eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$.]

**See next page !!!**

**5. Bonus: Measurement** (6 points*)

In the basis $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$ the Hamiltonian $\hat{H}$ and the observable $\hat{A}$ is given by

$$\hat{H} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \qquad \hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4}$$

We prepare the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|v_1\rangle + |v_3\rangle\big). \tag{5}$$

(a) [2 points] Determine the expectation value for the energy $\langle\hat{H}\rangle$ and the standard deviation $\Delta E = \sqrt{\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2}$.

(b) [2 points] Given that the state (5) is initialized and we measure the observable $\hat{A}$. Which values $a_i$ will be measured and with which probabilities $P(a_i)$ ? What are the corresponding states after the measurement?

(c) [2 points] Right after the measurement in (b) with the result $a_i$ the energy $\hat{H}$ is measured. Determine the values $E_j$ and the corresponding conditional probabilities $P_{a_i}(E_j)$ of the measurement. Determine further the joint probabilities $P(E_j, a_i) = P_{a_i}(E_j)P(a_i)$.