Übungen zur Modernen Theoretischen Physik I SS 14

Prof. Dr. Gerd Schön
Andreas Heimes, Dr. Andreas Poenicke

English Sheet – Blatt 9
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1. Harmonic Oscillator and Angular Momentum (4 Points)

We investigate the two-dimensional harmonic oscillator

$$H = \sum_{j=x,y} \frac{P_j^2}{2m} + \frac{1}{2} m\omega^2 X_j^2.$$

We define the ladder-operators $b_j^\dagger = \sqrt{\frac{m\omega}{2\hbar}} X_j - \frac{i}{\sqrt{2m\hbar\omega}} P_j$ and $b_j = \sqrt{\frac{m\omega}{2\hbar}} X_j + \frac{i}{\sqrt{2m\hbar\omega}} P_j$ which fulfill the commutator relation $[b_i, b_j^\dagger] = \delta_{ij}$. In the following we introduce the operators $a_+ = (b_x + ib_y)/\sqrt{2}$ and $a_- = (b_x - ib_y)/\sqrt{2}$.

(a) [1 Point] Show that a_+ and a_- fulfill the commutation-relations $[a_i, a_j^\dagger] = \delta_{ij}$. Furthermore, show that the tensor product $|n_+, n_-\rangle = |n_+\rangle \otimes |n_-\rangle$ of the eigenstates of $N_+ = a_+^\dagger a_+$ and $N_- = a_-^\dagger a_-$ are eigenstates of H. Determine the eigenenergies and their degeneracy.

(b) [1 Point] Show that the three operators

$$J_+ = \hbar a_+^\dagger a_-, \quad J_- = \hbar a_-^\dagger a_+, \quad J_z = \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-).$$

satisfy the algebra $[J_+, J_-] = 2\hbar J_z$, $[J_z, J_\pm] = \pm\hbar J_\pm$ (see sheet 8 exercise 3).

(c) [1 Point] Show that $[\boldsymbol{J}^2, H] = 0$ and $[J_z, H] = 0$, and that the eigenstates $|n_+, n_-\rangle$ can be represented using the quantum-numbers j and m of the operators for the angular momentum $\boldsymbol{J}^2$ and J_z, i.e.

$$|j, m\rangle = \frac{(a_+^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(a_-^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle . \tag{1}$$

[Hint: At first show that $\boldsymbol{J}^2 = \frac{N}{2}(\frac{N}{2}+1)$ with $N = N_+ + N_-$ and $J_z = (N_+ - N_-)/2$.]

(d) [1 Point] Show that for $m < j$

$$J_+ |j, m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle ,$$

by applying J_+ to Eq. (1) and by using the commutator-relations for $a_\pm$ and $a_\pm^\dagger$. Moreover show that

$$K_+ |j, m\rangle = \hbar\sqrt{(j+m+1)(j-m+1)}\, |j+1, m\rangle ,$$

where $K_+ = \hbar\, a_+^\dagger a_-^\dagger$.

[Hint: First show that $a_\pm (a_\pm^\dagger)^p = (a_\pm^\dagger)^p a_\pm + p(a_\pm^\dagger)^{p-1}$ with $p \in \mathbb{N}$.]

2. Fock-Darwin Spectrum (2 Points)

We consider the two-dimensional harmonic oscillator in a magnetic field, i.e.

$$H = \frac{(P_x + \frac{qB}{2}X_y)^2}{2m} + \frac{(P_y - \frac{qB}{2}X_x)^2}{2m} + \frac{1}{2}m\omega^2(X_x^2 + X_y^2). \tag{2}$$

where we used the gauge $\boldsymbol{A} = B/2(-X_y, X_x, 0)$ for the vector potential. Express Eq. (2) in terms of the operators $a_\pm^\dagger$ and $a_\pm$ that have been defined in exercise 1. Determine and sketch the eigenenergies as a function of magnetic field.

3. Radial component of the Hydrogen Atom (4 Points)

Starting from the differential equation for the radial component

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} - \frac{e^2}{r}\right] u_{k,l}(r) = E_{k,l}u_{k,l}(r), \tag{3}$$

which has been derived in the lecture, we want to retrieve the solutions $u_{k,l}(r)$ as follows.

(a) [0.5 Points] Write Eq. (3) in terms of the dimensionless variable $\rho = 2\kappa r$ and the dimensionless parameter $\frac{1}{\lambda_{k,l}} = \frac{1}{\kappa a_0}$, where $\kappa = \sqrt{\frac{-2mE_{k,l}}{\hbar^2}}$ and $a_0 = \frac{\hbar^2}{me^2}$.

(b) [0.5 Points] Show that in the limit $\rho \to \infty$ the physically relevant solution is approximately given by $u_{k,l}(\rho) = \exp(-\rho/2)$.

(c) [1 Point] Make the ansatz $u_{k,l}(\rho) = \rho^{l+1}e^{-\rho/2}v_{k,l}(\rho)$ and show that $v_{k,l}$ fulfills the differential equation

$$\left[\rho\frac{d^2}{d\rho^2} + (2l+2-\rho)\frac{d}{d\rho} - \left(l+1-\frac{1}{\lambda_{k,l}}\right)\right] v_{k,l}(\rho) = 0\,.$$

(d) [1 Point] Solve the differential equation via a series expansion $v_{k,l}(\rho) = \sum_{p=0}^{\infty} b_p\rho^p$ and show that the coefficients b_p fulfill the recursive relation

$$p(2l+1+p)b_p = \left(l+p-\frac{1}{\lambda_{k,l}}\right)b_{p-1}$$

with $b_0 = 1$.

(e) [1 Point] In order for $u_{k,l}(\rho)$ to be a physically relevant solution the series expansion has to terminate at a particular value $p = k$ with $k = 1, 2, ...$. Find the condition that $\lambda_{k,l}$ has to fulfill, so that $b_k = 0$. After that, derive the eigenenergies $E_{k,l}$. Determine $u_{k,l}(r)$ for $\{k = 1,\, l = 0\}$, $\{k = 2,\, l = 0\}$ and $\{k = 1,\, l = 1\}$ and sketch the functions $\frac{u_{k,l}(r)}{r}$.