KARLSRUHER INSTITUT FÜR TECHNOLOGIE
INSTITUT F"UR THEORETISCHE FESTKÖRPERPHYSIK

Übungen zur Modernen Theoretischen Physik I - English Sheet **SS14**

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1. Particle in a magnetic field - Landau levels (2 Points)

Consider a particle with charge q in a homogeneous magnetic field $\mathbf{B} = B\hat{e}_z$. A convenient choice for the vector potential $\mathbf{A}$ for this problem is the Landau gauge with $\mathbf{A} = Bx\hat{e}_y$.

In the following we assume that the motion of the particle is restricted to the xy-plane, like e.g. in a 2 dimensional electron gas. The Hamiltonian thus reads

$$\hat{H} = \frac{1}{2m}\big(\hat{\mathbf{P}} - q\mathbf{A}\big)^2 = \frac{1}{2m}\Big(\hat{P}_x^2 + \big(\hat{P}_y - qBx\big)^2\Big). \tag{1}$$

Goal of this exercise is to find the eigenfunctions and -energies of this problem.

(a) [0,5 points] Use $[\hat{H}, \hat{P}_y] = 0$ and the eigenfunctions of $\hat{P}_y$ to find a separation of variables form of the wave function $\psi(x, y)$.

(b) [1 point] Show that in this way the Schrödinger equation can be rewritten as the one of 1d harmonic oscillator.

(c) [0,5 points] Use the known solutions of the harmonic oscillator to retrieve the eigenenergies and eigenfunctions of the Hamiltonian (1). Find the characteristic frequency ω_c of the problem.

2. Harmonic oscillator (2 points)

Using the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, the Hamiltonian of an harmonic oscillator can be written as

$$\hat{H} = \hbar\omega\big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\big). \tag{2}$$

We consider a system which at time $t = 0$ is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big). \tag{3}$$

Here $|0\rangle$ is the ground state and $|1\rangle$ the first excited state. The state evolves in time as $|\psi(t)\rangle = \hat{U}(t)\,|\psi(0)\rangle$, i.e. by applying the time evolution operator $\hat{U}(t) = \exp\big(-i\hat{H}t/\hbar\big)$.

(a) [0,5 points] Compute the state $|\psi(t)\rangle$ for $t > 0$.

(b) [1 point] Compute $\langle\hat{X}\rangle(t) = \langle\psi(t)|\,\hat{X}\,|\psi(t)\rangle$ with $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\big(\hat{a}^\dagger + \hat{a}\big)$, and $\langle\hat{P}\rangle(t) = \langle\psi(t)|\,\hat{P}\,|\psi(t)\rangle$ with $\hat{P} = \mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\big(\hat{a}^\dagger - \hat{a}\big)$.

(c) [0,5 points] Compute the correlator $\langle\hat{X}_H(t)\hat{X}_H(0)\rangle$. Hint: Use the Heisenberg picture.

3. Properties of the angular momentum operator (3 points)

The vector operator $\hat{\mathbf{J}}$ with $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ defines an angular momentum operator if the following commutation relations apply:

$$[\hat{J}_x, \hat{J}_y] = \mathrm{i}\hbar\hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = \mathrm{i}\hbar\hat{J}_x, \quad \text{and} \quad [\hat{J}_z, \hat{J}_x] = \mathrm{i}\hbar\hat{J}_y \tag{4}$$

In addition to the vector components of the angular momentum operator $\hat{J}_{x/y/z}$ we often make use of the following operators

$$\hat{\mathbf{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \quad \hat{J}_+ = \hat{J}_x + \mathrm{i}\hat{J}_y, \quad \text{and} \quad \hat{J}_- = \hat{J}_x - \mathrm{i}\hat{J}_y. \tag{5}$$

Use the given relations and definitions to show:

(a) [1 point] $\quad [\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+, \quad [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_- \quad \text{und} \quad [\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z.$ (6)

(b) [1 point] $\quad [\hat{\mathbf{J}}^2, \hat{J}_z] = [\hat{\mathbf{J}}^2, \hat{J}_+] = [\hat{\mathbf{J}}^2, \hat{J}_-] = 0.$ (7)

(c) [1 point]

$$\begin{aligned} \hat{J}_+\hat{J}_- &= \hat{J}_x^2 + \hat{J}_y^2 + \hbar J_z = \hat{\mathbf{J}}^2 - \hat{J}_z^2 + \hbar\hat{J}_z \\ \hat{J}_-\hat{J}_+ &= \hat{J}_x^2 + \hat{J}_y^2 - \hbar J_z = \hat{\mathbf{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \\ \hat{\mathbf{J}}^2 &= \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z^2 \end{aligned} \tag{8}$$

4. Orbital angular momentum (3 points)

The orbital angular momentum operator is given by $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z) = \hat{\mathbf{R}} \times \hat{\mathbf{P}}$.

In spherical coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \quad \text{mit} \quad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

the gradient reads

$$\nabla_{r,\theta,\phi} = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \tag{9}$$

with

$$\begin{aligned} \hat{\mathbf{e}}_r &= \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\,\hat{\mathbf{e}}_x + \cos\theta\sin\phi\,\hat{\mathbf{e}}_y - \sin\theta\,\hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\phi &= -\sin\phi\,\hat{\mathbf{e}}_x + \cos\phi\,\hat{\mathbf{e}}_y. \end{aligned} \tag{10}$$

(a) [1 point] Show, using the sperical coordinates, that the angular momentum operator can be written as

$$\hat{L}_x = \frac{\hbar}{i}\Big(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big), \quad \hat{L}_y = \frac{\hbar}{i}\Big(\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi}\Big) \quad \text{und} \quad \hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{11}$$

(b) [1 point] Consider a particle in a state

$$\psi(\mathbf{r}) = (x + y + 2z)Ne^{-r^2/\alpha^2} \tag{12}$$

with $N, \alpha \in \mathbb{R}$. Using

$$\hat{\mathbf{L}}^2 = -\hbar^2\Big(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta}\Big), \tag{13}$$

show that $\psi(\mathbf{r})$ is an eigenfunction of $\hat{\mathbf{L}}^2$,

$$\hat{\mathbf{L}}^2\psi(\mathbf{r}) = l(l+1)\hbar^2\psi(\mathbf{r}), \tag{14}$$

and identify the value l.

(c) [1 point] Write the wave function (12) as a superposition of appropriate spherical harmonics. Which values can be measured for $\hat{L}_z$? What are the probabilities to find these values?