Übungen zur Modernen Theoretischen Physik I – English Sheet SS 14

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1. Time-Evolution (4 Points)

We consider the Hamiltonian

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z \tag{1}$$

in the basis of the two states $\{|0\rangle, |1\rangle\}$. At the time instant $t = 0$ the state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2}$$

with $|\alpha|^2 + |\beta|^2 = 1$ is initialized. Furthermore $\hat{\sigma}_i$ $(i = x, y, z)$ are the Pauli-matrices of sheet 6 exercise 3.

(a) [1 Point] Determine $|\psi(t)\rangle$ for times $t > 0$ and calculate $\langle\psi(t)|\hat{\sigma}_y|\psi(t)\rangle$.

(b) [1 Point] Calculate $\hat{\sigma}_y^H(t)$ in the Heisenberg-picture and derive $\langle\psi(0)|\hat{\sigma}_y^H(t)|\psi(0)\rangle$. Compare your result with the one you obtained in (a).

(c) [1 Point] It is given that at time instant $t = 0$ the state (2) is prepared. At $t = \tau_1$ the observable $\hat{\sigma}_x$ is measured. What are the possible values and the corresponding probabilities?

(d) [1 Point] After the measurement in (c) we measure $\hat{\sigma}_x$ again at a later time $\tau_2 > \tau_1$. What are the possible values and the corresponding probabilities?

2. Probability-Distribution and characteristic Function (3 Points)

The so-called characteristic function is defined by

$$F(z) = \langle e^{iz\hat{A}}\rangle\,, \tag{3}$$

where $\langle\cdots\rangle$ is the expectation value corresponding to the state $|\psi\rangle$.

(a) [1 Point] Show that the expectation values of arbitrary powers of $\hat{A}$, i.e. $\langle\hat{A}^n\rangle$ with $n \in \mathbb{N}$, can be obtained by differentiating with respect to z.

(b) [1 Point] Show that the probability to measure a_i during the measurement of $\hat{A}$ is given by

$$P(a_i) = \int \tfrac{dz}{2\pi} e^{-iza_i} F(z). \tag{4}$$

(c) [1 Point] A two-level-system $\{|0\rangle, |1\rangle\}$ is prepared in the ground-state $|\psi\rangle = |0\rangle$ and the observable $\hat{A} = \hat{\sigma}_x$ with the possible values a_m is measured. Derive the characteristic function and calculate $P(a_m)$.

3. Virial-Theorem (3 Points)

Similar to classical mechanics the Virial-theorem gives a relation between the expectation value of the kinetic energy $\langle \hat{T} \rangle$ and the expectation value of the potential energy $\langle V \rangle$. For stationary states

$$\left\langle \hat{T} \right\rangle = \left\langle \frac{\hat{\mathbf{P}}^2}{2m} \right\rangle = \frac{1}{2} \left\langle \hat{\mathbf{X}} \, \nabla V(\hat{\mathbf{X}}) \right\rangle . \tag{5}$$

This relation will be derived in the following subexercises:

(a) [1 Point] Show that in one dimension and for a stationary state

$$\langle [\hat{H}, \hat{X}\hat{P}] \rangle = \langle \hat{H}\hat{X}\hat{P} - \hat{X}\hat{P}\hat{H} \rangle = 0, \tag{6}$$

by directly calculating the expectation value of the right hand side of (6)).

(b) [1 Point] Then first calculate the commutator in (6), before you evaluate the expectation value, in order to prove the Virial-theorem (5).

(c) [1 Point] The potential $V(\hat{X})$ is given by $V(\hat{X}) = \lambda \hat{X}^n, \; n \in \mathbb{R}$.
Derive a relation between $\langle \hat{T} \rangle$ and $\langle V \rangle$.
Apply the Virial-theorem for the case of a harmonic oscillator in order to calculate the quotient of the expectation values for the kinetic energy and potential energy, $\langle \hat{T} \rangle \, / \, \langle V \rangle$.