Karlsruher Institut für Technologie

Institut für Theoretische Festkörperphysik

Übungen zur Modernen Theoretischen Physik I
– English Sheet –

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Sheet 6

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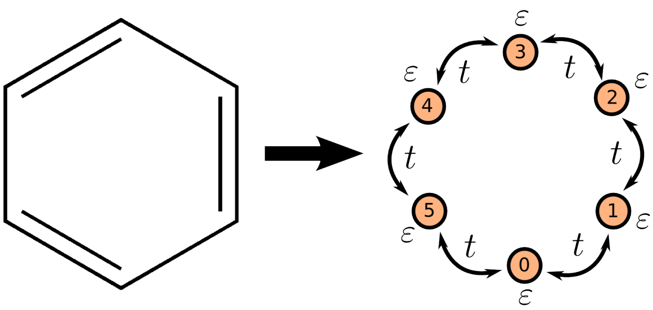
1. Benzene (3 points)

A benzene ring is composed of six carbon atoms. This system can be modeled as single-particle levels with energy ε coupled by a hopping amplitude t. In the local basis $\{|n\rangle\} = \{|0\rangle, |1\rangle, ..., |5\rangle\}$, where $|n\rangle$ denotes the localized state on the nth atom, the Hamiltonian reads

$$\hat{H} = t\sum_{n=0}^{5}\big(|n+1\rangle\langle n| + |n\rangle\langle n+1|\big) + \varepsilon\sum_{n=0}^{5}|n\rangle\langle n|\ ,$$

with the periodic boundary condition $|0\rangle = |6\rangle$. Determine the eigenvalues and eigenvectors in the basis $\{|n\rangle\}$.

[Hint: Diagonalize the Hamiltonian, using the Fourier representation, i.e. $|k\rangle = \frac{1}{\sqrt{6}}\sum_{n=0}^{5} e^{ikn}|n\rangle$.]



2. Baker-Hausdorff Formula (2 points)

It is given that $\hat{A}$ and $\hat{B}$ commute with the commutator $[\hat{A},\hat{B}]$, i.e. $[\hat{A},[\hat{A},\hat{B}]] = 0$ and $[\hat{B},[\hat{A},\hat{B}]] = 0$. Show, that in this case

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}.$$

[Hint: Define an operator $\hat{T}(\lambda) := e^{\hat{A}\lambda}e^{\hat{B}\lambda}$ and consider $\frac{\partial\hat{T}(\lambda)}{\partial\lambda}$. Use the relation $[\hat{B},\hat{A}^n] = n\hat{A}^{n-1}[\hat{B},\hat{A}]$ (s. Sheet 5, Ex. 3d) to calculate the commutator $[\hat{B},e^{-\hat{A}\lambda}]$.]

3. Measurement (5 Punkte)

A qubit (**qu**antum **bit**) is a quantum-mechanical two-level system. We have seen an example in exercise 2 c) of sheet three, where we were discussing the double-well potential. In the basis of the energetically lowest eigen-states of this system, $\{|1\rangle, |2\rangle\}$, the Hamiltonian can be written as

$$\hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \frac{E_1 + E_2}{2}\mathbb{1} + \frac{E_1 - E_2}{2}\hat{\sigma}_z = \varepsilon\mathbb{1} - \frac{\delta\varepsilon}{2}\hat{\sigma}_z, \tag{1}$$

where $\mathbb{1}$ is the 2×2-unit matrix and

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

are the Pauli-matrices.

It is given that the qubit is initialized in the state $|\psi\rangle = \alpha|1\rangle + \beta|2\rangle$.

(a) [1 point] Calculate the expectation value for the energy $\langle\hat{H}\rangle$ and the standard-deviation $\Delta E = \sqrt{\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2}$.

(b) [1 point] Now we measure the observable $\hat{A} = \hat{\sigma}_x$. What values can be measured and what is the corresponding probability? What is the corresponding state right after the measurement?

(c) [1 point] Right after the measurement in (b) the energy $\hat{H}$ is measured. Again determine the measurement-value and the corresponding probability.

(d) [2 points] Now the qubit will be initialized in the ground-state $|1\rangle$. The observables $\hat{B} = \hat{\sigma}_y$ and $\hat{A} = \hat{\sigma}_x$ will be measured one right after the other in the order $\hat{B}$ then $\hat{A}$. What are the possible measurement results and what are the corresponding probabilities?