"Ubungen zur Modernen Theoretischen Physik I **SS14**
- English Sheet -

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Blatt 5
Besprechung 28.05.2014

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1. Linear and quadratic Stark effect (2.5 points)

We focus on the two energetically lowest eigenstates $|1\rangle$ und $|2\rangle$ of the double-well potential of exercise 2c) sheet 3:

$$\left\langle x \,\middle|\, \hat{\mathcal{H}}_0 \,\middle|\, x \right\rangle = H_0(x) = -\frac{\hbar^2}{2m}\partial_x^2 + V(x) \qquad \text{mit} \qquad V(x) = \begin{cases} \infty & |x| > b \\ v_0\delta(x) & |x| \le b. \end{cases}$$

In position representation these states are given by

$$\psi_i(x) \equiv \langle x|i\rangle = \begin{cases} A_i \sin(k_i[b+x]) & -b < x < 0 \\ B_i \sin(k_i[b-x]) & 0 \le x < b \\ 0 & \text{otherwise} \end{cases}$$

with $A_1 = B_1$ and $A_2 = -B_2$. (Here, it is not necessary to calculate the constant wave numbers k_1 and k_2.)

(a) [0.5 points] Determine the coefficients A_i by the normalization condition $\langle i|i\rangle = 1$ for $i = 1, 2$.

(b) [1 point] Now we consider an applied electric field $\vec{E} = \mathcal{E}\hat{e}_x$. The Hamiltonian is given by

$$\langle x|\hat{\mathcal{H}}|x\rangle = H(x) = H_0(x) + q\mathcal{E}x. \tag{1}$$

Calculate the matrix elements $h_{ij} = \langle i|\hat{\mathcal{H}}|j\rangle$, $i, j = 1, 2$ and show that in the basis

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

the Hamiltonian can be written as

$$\hat{\mathcal{H}} = \begin{pmatrix} E_1 & d\,\mathcal{E} \\ d^*\,\mathcal{E} & E_2 \end{pmatrix}. \tag{3}$$

Hints:

- Use that $\psi_{1,2}(x)$ are eigenfunctions of $H_0(x)$ with eigenvalues $E_{1,2}$.
- It is not necessary to evaluate the integral $\int_0^b \sin(k_1x)\sin(k_2x)x\mathrm{d}x$. Replace it by a constant $C(k_1, k_2)$.

(c) [1 point] Calculate the eigenvalues and (unnormalized) eigenvectors of $\hat{\mathcal{H}}$.
Evaluate the eigenvalues in the limit of:

(i) weak field $|d|\mathcal{E} \ll E_2 - E_1$ [Hint: $\sqrt{1+x} = 1 + \frac{x}{2} + \cdots$],

(ii) strong field $|d|\mathcal{E} \gg E_2 - E_1$.

2. Functions of operators (2.5 points)

(a) [0.5 points] Calculate $e^{\hat{\sigma}_z}$, with $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) [1 point] Show that in case of a time-independent operator $\hat{A}$

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{A}t} = \hat{A} e^{\hat{A}t} = e^{\hat{A}t} \hat{A}.$$

Additionally calculate $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\hat{A}t} e^{\hat{B}t}\right)$, with $\hat{A}$ and $\hat{B}$ beeing time-independent.

(c) [1 point] Show that for an arbitrary time-dependent operator $\hat{A}(t)$ in general

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{A}(t)} \neq \frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t} e^{\hat{A}(t)}. \tag{4}$$

Which is the condition for both sides of Eq. 4 to be equal?

3. Hermitian conjugates and commutator algebra (3 points)

(a) [0.5 points] Calculate the Hermitian adjoint of $\hat{X}\hat{P}_x$ and $i[\hat{X}^2, \hat{P}_x]$.

(b) [1 point] Given that $\hat{G}$ is Hermitian, show that for $\hat{F} = e^{i\hat{G}}$ the relation $\hat{F}^\dagger = \hat{F}^{-1}$ holds.

(c) [0.5 points] Show that $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

(d) [1 point] Prove, that in case $[[A, B], A] = 0$ the relation $[\hat{A}^n, \hat{B}] = n\hat{A}^{n-1}[\hat{A}, \hat{B}]$ is true ($n \in \mathbb{N}$).
With that, calculate $[\hat{P}, \hat{X}^n]$.

4. Position and momentum representation (2 points)

Starting in momemtum representation

$$\hat{P}\,|p\rangle = p\,|p\rangle \quad \text{with} \quad u_p(x) \equiv \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}, \tag{5}$$

show that in position representation the momentum operator $\hat{P}$ can be written as

$$\langle x|\hat{P}|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x).$$

Next week there will be the first exam:

Wednesday, 04.06, 14:00-16:00.

According to the first letter of the surname, the distribution is as follows,

A-R: Gerthsen-Hörsaal
S-Z: Gaede-Hörsaal

Please bring your student-ID. It is not necessary to register in order to attend. You are allowed to bring a double-sided and handwritten DIN A4 paper to the exam.