"Ubungen zur Modernen Theoretischen Physik I SS14
- English Sheet -

Prof. Dr. Gerd Schön
Dr. Andreas Poenicke, Andreas Heimes

Blatt 4
Besprechung 21.05.2014

---

1. Particle in a triangular potential well (3 points)

In this exercise we investigate a particle in a triangular potential of the form

$$V(x) = \begin{cases} \infty, & \text{for} \quad x \leq 0 \\ ax, & \text{for} \quad x > 0. \end{cases}$$

To solve the problem we make use of the Airy functions. The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are solutions to the differential equation

$$f''(x) - xf(x) = 0. \tag{1}$$

They are depicted in figure 1.

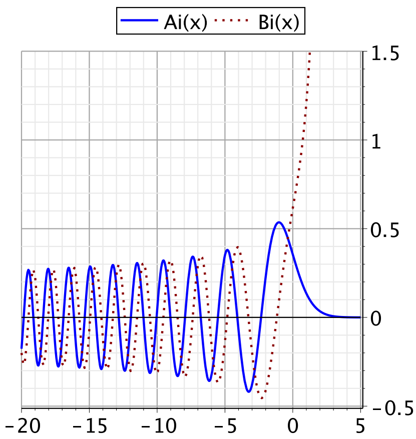

Figure 1:

For $x \to \infty$ $\mathrm{Ai}(x)$ is decreasing exponentially to zero while $\mathrm{Bi}(x)$ is increasing exponentially[1]. The roots of these functions are listed e.g. in Abramowitz & Stegun.

The objective is to obatin expressions for the wave functions and energy eigenvalues. To this end:

(a) [1 point] Transform the Schrödinger equation by the change of variables $\bar{x} = x - \frac{E}{a}$ and write the equation in a dimensionless representation by introducing a characteristic length scale.

(b) [1 point] Find the energy levels assuming that the roots x_n of the Airy functions are known.

(c) [1 point] Sketch the three wave functions with the lowest energy.

---

[1] The asymtotic formulas are

$$\mathrm{Ai}(x) \to \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}, \quad \mathrm{Bi}(x) \to \frac{e^{\frac{2}{3}x^{3/2}}}{\sqrt{\pi}\,x^{1/4}} \quad \text{for } x \to \infty,$$

but not required in this exercise.

2. General properties of one-dimensional systems (2 points)

We consider one-dimensional systems, but without explicitely solving the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \Big[V(x) - E\Big]\psi = 0. \tag{2}$$

(a) **Degeneracy** [1 Point]
Show that bound states are non degenerate.
Assume two functions $\psi_1(x)$ and $\psi_2(x)$ to be eigenfunctions of the Schröedinger equation Eq. 2 with the same eigenvalue E_1. Considering the properties of bound states for $|x| \to \infty$ calculate $\psi_1'\psi_2 - \psi_2'\psi_1$ and show that ψ_1 and ψ_2 are linearly dependent.

(b) **Number of nodes** [1 Point]
If the eigenfunctions are sorted in ascending order according to their energy ($E_0 < E_1 < E_2 \dots < E_n < \dots$) where E_n is the energy eigenvalue of ψ_n, then it can be shown that the n-th eigenfunction ψ_n has exactly n nodes[2].
As a first step one can prove that ψ_{n+1} has more nodes than ψ_n:
Show that if ψ_n has two neighboring nodes at x_1 and x_2, ψ_{n+1} has to have at least one node in this interval.
(Hint: Calculate $\psi_n'\psi_{n+1} - \psi_n\psi_{n+1}'\big|_a^b$ in a suitable interval $[a, b]$.)

3. Harmonic oscillator (2 Points)

It is possible to estimate the ground state energy of an harmonic oscillator solely by the uncertainty principle and the potential

$$U(x) = \frac{1}{2}m\omega^2x^2. \tag{3}$$

Using only the uncertainty relation for Δx and Δp and the potential $U(x)$ give a lower bound for the expectation value of the Hamilton operator $\langle\hat{H}\rangle$. Compare this estimate with the known exact solution for the ground state energy E_0.

4. Particle in a ring (3 Punkte)

The motion of a free particle ($U(x) = 0$) is restricted to a ring of radius R. The wavefunction has to be single-valued along the closed trajectory thus the boundary condition is $\psi(\phi+2\pi) = \psi(\phi)$.

(a) [1 point] Write the Hamilton operator $\hat{H}$ in polar coordinates and retrive its eigenfunctions. Show that, as a consequence of the boundary condition, the energies are quantized and give an expression for the energy eigenvalues. What is the degeneracy of the energy levels?

(b) [1 point] Normalize the wavefunction and show that the probability density $|\psi(\phi)|^2$ is constant.

(c) [1 point] The angular momentum operator $\hat{L}_z$ is given as

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{4}$$

Show that the wave function $\psi(\phi)$ is also an eigenfunction of $\hat{L}_z$ and compute the eigenvalues.

---

[2] Nodes at the boundaries of the confining interval are not counted