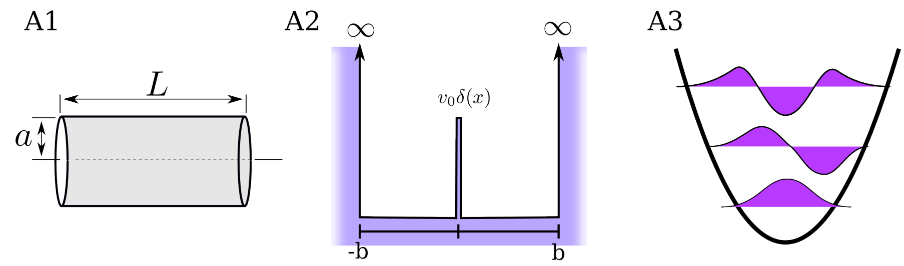
Karlsruher Institut für Technologie — Institut für Theoretische Festkörperphysik

Übungen zur Modernen Theoretischen Physik I SS14
– English Sheet –

Prof. Dr. Gerd Schön
Andreas Heimes, Dr. Andreas Poenicke

Blatt 3
Besprechung 14.05.2014



1. Particle in a Cylinder (3 Points)

A particle lives in a three-dimensional potential (Fig. A1)

$$V(r,\varphi,z) = \begin{cases} 0 & \text{for } r < a \text{ and } 0 \le z < L \\ \infty & \text{otherwise} \end{cases}$$

The Schrödinger-equation in cylindrical-coordinates is given by

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\partial_r\left(r\partial_r\psi\right) + \frac{1}{r^2}\partial_\phi^2\psi + \partial_z^2\psi\right] + V\psi = E\psi.$$

Solve with the ansatz $\psi(r,\varphi,z) = A\exp(in\varphi)\sin(k_z z)R(r)$:

(a) (1 Punkt) Which conditions have to be fulfilled for n and k_z?

(b) (1 Punkt) Show that $R(r) = J_n(kr)$ with $k = \sqrt{\frac{2mE}{\hbar^2} - k_z^2}$, where J_n is the Bessel-function of the first kind. What condition do you obtain for the energy-quantization?

(c) (1 Punkt) Show that in the limit $a \ll L$ the lowest energy levels are given by $E_l = \frac{\hbar^2\gamma^2}{2ma^2} + \frac{\hbar^2\pi^2}{2mL^2}l^2$, with $l = 1, 2, 3, ...$ and γ being the first zero of $J_0(x)$.

2. Double-Well Potential (4 Points)

In this exercise we discuss the double-well potential in figure A2,

$$V(x) = \begin{cases} \infty & |x| > b \\ v_0\delta(x) & |x| \le b \end{cases}$$

with $v_0 > 0$.

(a) (1 Point) Given the coordinate-dependent function g, the parity operator P is defined by

$$Pg(x) = g(-x)$$
$$P^2 g(x) = g(x).$$

Show that:

(i) If $H(-x) = H(x)$ and $\psi(x)$ is a solution of the Schrödinger-equation, show that $P\psi(x)$ is a solution as well.

(ii) The eigenfunctions of P are either symmetric or anti-symmetric, i.e. $\psi_{s/a}(-x) = \pm\psi_{s/a}$.

(b) (1 Point) Make an appropriate ansatz for $\psi_{s/a}$, which fulfills the matching-conditions at $|x| = 0$ and $|x| = b$ and show that the energy-quantization in the symmetric and anti-symmetric case is correspondingly given by

$$\frac{1}{k}\tan(bk) = -\frac{\hbar^2}{mv_0} \qquad \text{and} \qquad \sin\big(kb\big) = 0\,, \tag{1}$$

with $\hbar k = \sqrt{2mE}$.

(c) (1 Point) Discuss the limit $v_0 \gg \frac{\hbar^2}{mb}$ and calculate approximately the energy-splitting $\Delta E = E_2 - E_1$ of the two energetically lowest eigenstates ψ_1 and ψ_2, which should be given by

$$\Delta E = \frac{\hbar^4\pi^2}{m^2 b^3 v_0}.$$

[Hint: Use the Taylor-expansion of $\tan(x)$ around $x = \pi$.]

(d) (1 Point) At the time instant $t = 0$, the state ψ is a superposition of the eigenstates ψ_1 and ψ_2 that we have discussed in (c),

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}\big(\psi_1(x, t = 0) + \psi_2(x, t = 0)\big)\,.$$

Discuss qualitatively the time-evolution of $|\psi(x,t)|^2$ for $t > 0$ by solving the time-dependent Schrödinger-equation.

3. Hermite Polynomials (3 Points)

In the lecture, the eigenfunctions of the harmonic oscillator were discussed. In figure A3 you find a sketch of these. They are linked to the so called Hermite polynomials

$$H_n(z) = (-1)^n e^{z^2}\partial_z^n e^{-z^2}, \qquad n \geq 0$$

(a) (1 Point) First, show that the function e^{-t^2+2zt} is the generating function of the Hermite polynomials, i.e.

$$e^{-t^2+2zt} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(z). \tag{2}$$

[Hint: Use the Taylor-expansion of $e^{-(z-t)^2}$ around $t = 0$.]

(b) (1 Point) Derive with the help of (2) the following recursion relations for H_n:

$$\partial_z H_n(z) = 2n\, H_{n-1}(z), \qquad n \geq 1 \tag{3}$$

and

$$H_{n+1}(z) = 2z\, H_n(z) - 2nH_{n-1}(z), \qquad n \geq 1 \tag{4}$$

Use (3) and (4) to derive the differential equation

$$\big[\partial_z^2 - 2z\partial_z + 2n\big]H_n(z) = 0. \tag{5}$$

[Hint: Eqs. (3) and (4) can be proven by differentiating (2) with respect to z or t.]

(c) (1 Point) Prove the orthogonality condition for the Hermite polynomials,

$$\int_{-\infty}^{\infty} dz\, e^{-z^2} H_n(z)H_m(z) = 0, \qquad \text{für } n \neq m \tag{6}$$

[Hint: Multiply the left hand side of (5) with $H_m(z)e^{-z^2}$ and integrate over z. Subtract the corresponding equation with m and n exchanged.]