Karlsruher Institut für Technologie | Institut für Theoretische Festkörperphysik

Übungen zur Modernen Theoretischen Physik I - English Sheet **SS14**

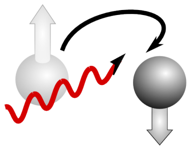
Prof. Dr. Gerd Schön | **Blatt 10**

Andreas Heimes, Dr. Andreas Poenicke | **Besprechung 09.07.2014**

---

1. Spin in an electromagnetic field (4 points)

We consider a spin-$\frac{1}{2}$ with a Zeeman splitting $\hbar\omega$, resonantly coupled to an electromagnetic mode of the same frequency ω. The Hamiltonian of this system is given by



$$\hat{H} = -\frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + g(\hat{a}\hat{\sigma}_- + \hat{a}^\dagger\hat{\sigma}_+). \quad (1)$$

.

The first term represents the Zeeman splitting, with the ground state $|\uparrow\rangle$ and ground state energy $-\frac{\hbar\omega}{2}$ and an excited state $|\downarrow\rangle$ with energy $\frac{\hbar\omega}{2}$. The second term describes the monochromatic electromagnetic field with creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, and eigenstates $|n\rangle$, where $\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle$. The last term of (1) is the coupling of the spin to the electromagnetic mode. The matrices $\hat{\sigma}_\pm$ are given by $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$ and satisfy

$$\sigma_+\,|\downarrow\rangle = |\uparrow\rangle\,,\quad \sigma_-\,|\uparrow\rangle = |\downarrow\rangle\,,\quad \sigma_+\,|\uparrow\rangle = 0,\quad \sigma_-\,|\downarrow\rangle = 0. \quad (2)$$

The Hilbert space of the system is spanned by the states $|\uparrow\rangle\{|n\rangle\}$ and $|\downarrow\rangle\{|n\rangle\}$ with $\{|n\rangle\} = \{|0\rangle\,, |1\rangle\,, |2\rangle\,, ...\}$.

(a) [2 points]
Write the Hamiltonian (1) in the basis

$$\{|\uparrow\rangle\,|0\rangle\,, |\uparrow\rangle\,|1\rangle\,, |\downarrow\rangle\,|0\rangle\,, |\uparrow\rangle\,|2\rangle\,, |\downarrow\rangle\,|1\rangle\,, ...\,, |\uparrow\rangle\,|n+1\rangle\,, |\downarrow\rangle\,|n\rangle\,, ...\}$$

and show, that it can be written as

$$\hat{H} = \begin{pmatrix} 0 & 0 & & & & \\ 0 & \hat{H}_1 & \hat{\mathbf{0}} & & & \\ & \hat{\mathbf{0}} & \hat{H}_2 & \hat{\mathbf{0}} & & \\ & & \hat{\mathbf{0}} & \ddots & \hat{\mathbf{0}} & \\ & & & \hat{\mathbf{0}} & \hat{H}_n & \hat{\mathbf{0}} \\ & & & & \hat{\mathbf{0}} & \ddots \end{pmatrix}.$$

Here $\hat{H}_n$ are 2×2-matrices, $\hat{\mathbf{0}} \equiv 0\,\hat{\mathbb{1}}$ and $\mathbb{1}$ is the 2×2 unit matrix. Show that

$$H_{n+1} = \hbar\omega(n+1)\mathbb{1} + g\sqrt{n+1}\hat{\sigma}_x. \quad (3)$$

(b) [1 point] Show, that the number of excitations $\hat{N}_e = \hat{a}^\dagger\hat{a} + |\downarrow\rangle\langle\downarrow|$ is conserved, i.e. $i\hbar\frac{d\hat{N}_e^H}{dt} = -[\hat{H},\,\hat{N}_e^H] = 0$. Using this, interpret your findings of exercise 1a).

(c) [1 point] Determine the eigenenergies and eigenstates of the Hamiltonian (3).

2. Hydrogen atom in a classical radiation field (3 points)

Consider a hydrogen atom in a time dependent electric field. The interaction with the electric field is, in the dipole approximation, given by

$$H'(t) = e\boldsymbol{E} \cdot \hat{\boldsymbol{R}} \cos(\omega t).$$

In the following we want to examine the transitions from the ground state $|nlm\rangle = |100\rangle$ of the hydrogen atom, to the the exited states $|2lm\rangle$.

To compute the transition rates $\Gamma_{|100\rangle\to|2lm\rangle}$ we use (without derivation) Fermi's golden rule:

$$\Gamma_{|100\rangle\to|2lm\rangle} = \frac{2\pi}{\hbar} |\langle 2lm|e\boldsymbol{E} \cdot \hat{\boldsymbol{R}}|100\rangle|^2 \delta(E_{2lm} - E_{100} - \hbar\omega). \tag{4}$$

Here E_{nlm} are the eigenenergies of the hydrogen atom, whose eigenfunctions are given by $\langle r\theta\varphi|nlm\rangle = R_{nl}(r)Y_l^m(\theta, \varphi)$.

As a first step, parametrize the interaction term

$$\boldsymbol{E} \cdot \boldsymbol{r} = \left[E_x r \sin(\theta)\cos(\varphi) + E_y r \sin(\theta)\sin(\varphi) + E_z r \cos(\theta)\right] \tag{5}$$

using spherical harmonics. Now calculate all non-vanishing matrix elements $\langle 2lm|e\boldsymbol{E} \cdot \hat{\boldsymbol{R}}|100\rangle$ and, with these compute, $\Gamma_{|100\rangle\to|2lm\rangle}$.

[Hint: Use the relation $\int_0^\infty dr r^3 R_{21}(r)R_{10}(r) = \frac{a_0}{\sqrt{6}}\frac{2^8}{3^4}$ (a_0 is the Bohr radius).]

3. Stern-Gerlach experiment with precession (3 points)

Now we study a Stern-Gerlach experiment with two Stern-Gerlach apparatuses in sequence. In the first, the magnetic field points in z-direction, and the spin is prepared either in the state $|\uparrow\rangle$ or $|\downarrow\rangle$. In the second apparatus the magnetic field points in x-direction. Between these two apparatuses a homogenous magnetic field is applied in y-direction, giving rise to a precession of the spin during the time of flight T from the first to the second apparatus. Calculate the intensity of the points which will be detected on a screen behind the setup.