1. Warm-Up (10 points)

(a) **Translation** [2 points]

Show that the state $|\phi\rangle = e^{iPa/\hbar}|\psi\rangle$ corresponds to the state $|\psi\rangle$ shifted by the distance a, i.e. $\phi(x) = \psi(x+a)$. Here P is the momentum operator.

(b) **Harmonic Oscillator** [3 points]

Show that

$$e^{-i\alpha N} a^\dagger e^{i\alpha N} = e^{-i\alpha} a^\dagger, \tag{1}$$

where $a^\dagger$ and a are the ladder-operators of the harmonic oscillator with $[a, a^\dagger] = 1$ and $N = a^\dagger a$.

[Hint: Apply the left- and right-hand-site of (1) onto a general state $|\phi\rangle = \sum_n b_n |n\rangle$, where $N|n\rangle = n|n\rangle$ and $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$.]

(c) **Bloch-Equations** [3 points]

We consider a particle with magnetic moment $\boldsymbol{M} = \gamma \boldsymbol{L}$ in a magnetic field $\boldsymbol{B}$,

$$H = -\boldsymbol{M} \cdot \boldsymbol{B}.$$

Show with the help of the commutator relations for the angular momentum $\boldsymbol{L}$ that the expectation value $\langle \boldsymbol{M} \rangle$ satisfies the equation of motion

$$\frac{d}{dt}\langle \boldsymbol{M} \rangle = \gamma \langle \boldsymbol{M} \rangle \times \boldsymbol{B}.$$

[Hint: $[\boldsymbol{a} \times \boldsymbol{b}]_i = \sum_{jk} \varepsilon_{ijk} a_j b_k$.]

(d) **Stark-Effect** [2 points]

A two level system $\{|\psi_+\rangle, |\psi_-\rangle\}$ couples to a classical electric field $\mathcal{E}$. The Hamiltonian is given by

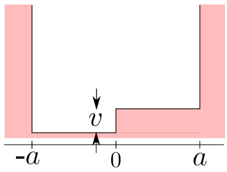
$$\begin{pmatrix} \langle\psi_+|H|\psi_+\rangle & \langle\psi_+|H|\psi_-\rangle \\ \langle\psi_-|H|\psi_+\rangle & \langle\psi_-|H|\psi_-\rangle \end{pmatrix} = \begin{pmatrix} E_+ & \alpha\mathcal{E} \\ (\alpha\mathcal{E})^* & E_- \end{pmatrix}.$$

Determine the eigenenergies and discuss the limits
$|\alpha\mathcal{E}| \gg |E_- - E_+|$ and $|\alpha\mathcal{E}| \ll |E_- - E_+|$.

[Hint: $\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots, \qquad (x \ll 1)$.]

2. Potential-Well (5 points)

We consider a particle in a one-dimensional potential well



$$V(x) = \begin{cases} \infty & x \le -a \\ v\,\theta(x) & -a < x \le a \\ \infty & a < x \end{cases},$$

where $\theta(x)$ is the Heaviside-Theta-function.

(a) [2 points] Make an appropriate ansatz for the wavefunction in the intervals $-a < x \le 0$ und $0 < x \le a$, which fulfills the matching conditions at $x = \pm a$.

(b) [3 points] Use the matching conditions at $x = 0$ to show that the eigenenergies are determined by the transcendent equation

$$q \tan(ka) = -k \tan(qa),$$

where $k = \sqrt{2mE/\hbar^2}$ and $q = \sqrt{2m(E - v)/\hbar^2}$.

3. Harmonic Oscillator (7 points)

Consider a one-dimensional oscillator,

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2, \quad \text{mit} \quad [X, P] = i\hbar.$$

We define the ladder operators $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - \frac{i}{\sqrt{2m\hbar\omega}}P$ and $a = \sqrt{\frac{m\omega}{2\hbar}}X + \frac{i}{\sqrt{2m\hbar\omega}}P$. At the time instant $t = 0$ the system is initialized in the state

$$|\phi(0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |n+2\rangle),$$

where $|n\rangle$ are the eigenstates of the Hamiltonian, i.e. $H|n\rangle = \hbar\omega(n + 1/2)|n\rangle$.

(a) [2 points] Determine $|\phi(t)\rangle$ for $t > 0$ and derive the expectation value $\langle H\rangle$, where $\langle\cdot\rangle = \langle\phi(t)|\cdot|\phi(t)\rangle$.

(b) [5 points] Determine the time-dependence of ΔX^2, where $\Delta X^2 = \langle X^2\rangle - \langle X\rangle^2$.

[Hint: Write X and X^2 in terms of a and $a^\dagger$, where $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a|n\rangle = \sqrt{n}\,|n-1\rangle$.]

4. Spin-$\frac{1}{2}$ (7 points)

The Hamiltonian of a Spin-$\frac{1}{2}$ system is given by

$$H = -\frac{\hbar\omega}{2}\frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) = -\frac{\hbar\omega}{2}\begin{pmatrix} 0 & e^{-i\pi/4} \\ e^{i\pi/4} & 0 \end{pmatrix}, \tag{2}$$

written in the eigenbasis of σ_z, i.e. $\{|\uparrow\rangle, |\downarrow\rangle\}$. At the time instant $t = 0$ we prepare the state

$$|\phi(0)\rangle = |\uparrow\rangle . \tag{3}$$

(a) [3 points] At the time instant $t = 0$ we measure H. Which values E are measured and what are the corresponding probabilities $P(E)$?

(b) [4 points] Once again, the state (3) is initialized at the time instant $t = 0$. At a later time τ we measure the observable $A = \sigma_z$. What are the values a that we measure and what are the corresponding probabilities $P(a)$?

5. Three-Atomic Molecule (6 points)

We consider an electron in a molecule, which consists of three atoms A, B, C. The (orthonormal) wave-functions that are localized around the cores are termed as $|\varphi_a\rangle$, $|\varphi_b\rangle$ and $|\varphi_c\rangle$. First, if we neglect the possibility that the electron can hop from one atom to the other the system is characterized by the Hamiltonian H_0, for which we have $H_0|\varphi_i\rangle = E_0|\varphi_i\rangle$ $(i = a, b, c)$. Now, the atoms are coupled by an additional operator V,

$$\begin{aligned} V|\varphi_a\rangle &= t|\varphi_b\rangle + t|\varphi_c\rangle \\ V|\varphi_b\rangle &= t|\varphi_a\rangle + t|\varphi_c\rangle \\ V|\varphi_c\rangle &= t|\varphi_a\rangle + t|\varphi_b\rangle \end{aligned}$$

(a) [1 point] Express $H = H_0 + V$ as a 3×3-matrix in the basis $\{|\varphi_a\rangle, |\varphi_b\rangle, |\varphi_c\rangle\}$.

(b) [2 points] We define the operator

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Calculate T^2 and T^3 and express H in terms of the unit matrix $\mathbb{1}$, T and T^2.

(c) [1 point] Show that the eigenvectors of T are eigenvectors of H.

(d) [2 points] Show with the help of (b) and (c) that the eigenvalues of H are given by $E_0 + 2t\cos(2\pi n/3)$ with $n = 0, 1, 2$.