Übungen zur Modernen Theoretischen Physik I (SS 14)
- English Sheet -

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Sheet 2
Besprechung 07.05.2014

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1. Norm and Continuity (2 points)

Given the Schrödinger-equation $i\hbar\partial_t\psi(x,t) = H(x,t)\psi(x,t)$ with the coordinate and time dependent Hamiltonian $H(x,t) = -\frac{\hbar^2}{2m}\partial_x^2 + V(x,t)$ and $V(x,t) = V^*(x,t)$.

(a) (1 point) An arbitrary state $\psi(x,t_0)$ is normalized at time t_0, i.e. $\int_{-\infty}^{\infty} dx\, |\psi(x,t_0)|^2 = 1$. Show that $\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2$ is time-independent.

(b) (1 point) Show that the probability density $\rho(x,t) = |\psi(x,t)|^2$ is locally conserved, this means that the continuity equation

$$\partial_t\rho(x,t) + \partial_x J(x,t) = 0$$

holds. Here the probability-current is defined as $J(x,t) = \frac{1}{m}\mathrm{Re}\big[\psi^*(x,t)\frac{\hbar}{i}\partial_x\psi(x,t)\big]$.

2. Wave-packet and uncertainty relation (3 points)

A particle can be described by a wave-packet

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, g(k) e^{i(kx-\omega_k t)},$$

with momentum-distribution

$$g(k) = \sqrt{a}\exp(-a^2k^2/4)/(2\pi)^{1/4},$$

where $\omega_k = \hbar k^2/2m$.

(a) (1 point) First focus on the time instant $t = 0$. Show that $\psi(x,0)$ is a Gaussian as well. What is the width of this function and how does it dependent on a?

(b) (1 point) The spread in coordinate and momentum is defined by $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ and $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$, where

$$\langle A\rangle(t) = \int_{-\infty}^{\infty} dx\, \psi^*(x,t)\, A\, \psi(x,t)$$

is the expectation value of the operator A. Show that for $t = 0$ the uncertainty relation $\Delta x\Delta p = \hbar/2$ holds.

(c) (1 point) Now calculate $\psi(x,t)$ for arbitrary t and discuss the evolution of $|\psi(x,t)|^2$ with time. What do you now obtain for $\Delta x\Delta p$?

3. Delta-potentials (5 points + 1 bonus-point)

In the following we solve the one-dimensional Schrödinger-equation for a particle with mass m and energy E in the potential $V(x)$:

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right]\psi(x) = E\psi(x)$$

(a) (3 points) First we discuss the single Delta-potential $V(x) = -v_0\delta(x)$ mit $v_0 > 0$.

(i) Derive, as you have already done in exercise 3(a) at the first sheet, the connecting conditions for ψ and $\partial_x\psi$ at $x = 0$.

(ii) Given that $E < 0$, show that there exists a bound-state and that its energy is given by $E = -mv_0^2/2\hbar^2$.

(iii) An incoming particle from the left with energy $E > 0$ is partly reflected and partly transmitted. Calculate the transmission coefficient and show that the transmitted wave picks up a phase. Discuss also the limit $E \to \infty$.

(b) (2 points) Now we consider the double Delta-potential $V(x) = -v_0[\delta(x + a/2) + \delta(x - a/2)]$ with $v_0 > 0$. Solve the one-dimensional Schrödinger-equation and show that

$$e^{-\kappa a} = \pm\left(1 - \frac{\hbar^2\kappa}{mv_0}\right), \tag{1}$$

where $E = -\frac{\hbar^2\kappa^2}{2m}$ is the energy of the particle and $\kappa \in \mathbb{R}$. Solve graphically the transcendent equation (1).

(c) *Resonant tunneling* (**1 bonus-point**):
Now $E > 0$ and $V(x) = v_0[\delta(x + a/2) + \delta(x - a/2)]$ with $v_0 > 0$. What condition for $k = \sqrt{2mE}/\hbar$ has to be fulfilled so that the reflection coefficient vanishes?