Aufgabe 1:

a Wie in der Vorlesung

$$\frac{\hbar}{2e} I_1 = N\mu_1 - K\mu_4 - (N-K)\mu_2$$

$$\frac{\hbar}{2e} I_2 = N\mu_2 - N\mu_1 = 0$$

$$\frac{\hbar}{2e} I_3 = K\mu_3 - K\mu_2$$

$$\frac{\hbar}{2e} I_4 = K\mu_4 - K\mu_3 = 0$$

$$\mu_4 = \mu_3 = 0 \qquad [\text{Def: } \mu_3 = 0]$$

$$\mu_2 = \mu_1 = eV \qquad [\text{Def: } \mu_1 := eV]$$

)

$$\frac{\hbar}{2e} = N\mu_1 - (N-K)\mu_2 = KeV$$

$$\frac{1}{1-p} = \left(1 - \frac{N-K}{N}\right)^{-1} = \left(\frac{N-N+K}{N}\right)^{-1} = \left(\frac{K}{N}\right)^{-1}$$

$$\Rightarrow R_H = \cancel{\frac{\hbar}{2e}} \frac{\hbar}{2e^2 N} \frac{1}{1-p} = \frac{\hbar}{2eK}$$

Aufgabe 2

a) ~~KV =~~

$$H-\mu N = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \left(\Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \Delta_k^* c_{-k\downarrow} c_{k\uparrow}\right)$$

Bogoliubov-Transformation:

$$c_{k\uparrow} = u_k^* \gamma_{k\uparrow} + v_k \gamma_{k\downarrow}^\dagger$$

$$c_{k\downarrow}^\dagger = -v_k^* \gamma_{k\uparrow} + u_k \gamma_{k\downarrow}^\dagger$$

$$\Rightarrow c_{k\uparrow}^\dagger c_{k\uparrow} = \left(u_k \gamma_{k\uparrow}^\dagger + v_k^* \gamma_{k\downarrow}\right)\left(u_k^* \gamma_{k\uparrow} + v_k \gamma_{k\downarrow}^\dagger\right)$$

$$= |u_k|^2 \gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} + |v_k|^2 \gamma_{k\downarrow} \gamma_{k\downarrow}^\dagger + u_k v_k \gamma_{k\uparrow}^\dagger \gamma_{k\downarrow}^\dagger + v_k^* u_k^* \gamma_{k\downarrow} \gamma_{k\uparrow}$$

$$c_{k\downarrow}^\dagger c_{k\downarrow} = \left(-v_k^* \gamma_{k\uparrow} + u_k \gamma_{k\downarrow}^\dagger\right)\left(-v_k \gamma_{k\uparrow}^\dagger + u_k^* \gamma_{k\downarrow}\right)$$

$$= |v_k|^2 \gamma_{k\uparrow} \gamma_{k\uparrow}^\dagger + |u_k|^2 \gamma_{k\downarrow}^\dagger \gamma_{k\downarrow} - v_k^* u_k^* \gamma_{k\uparrow} \gamma_{k\downarrow} - u_k v_k \gamma_{k\downarrow}^\dagger \gamma_{k\uparrow}^\dagger$$

Damit:

$$\sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} = \sum_k \xi_k \Big[\left(|u_k|^2 - |v_k|^2\right)\left(\gamma_{k\uparrow}^\dagger \gamma_{k\uparrow} + \gamma_{k\downarrow}^\dagger \gamma_{k\downarrow}\right) + 2|v_k|^2 + 2 u_k^* v_k^* \gamma_{k\downarrow} \gamma_{k\uparrow} + 2 u_k v_k \gamma_{k\uparrow}^\dagger \gamma_{k\downarrow}^\dagger \Big]$$

$$c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} = (u_k \gamma^\dagger_{k\uparrow} + v_k^* \gamma_{k\downarrow})(-v_k^* \gamma_{k\uparrow} + u_k \gamma^\dagger_{k\downarrow})$$

$$= -u_k v_k^* \gamma^\dagger_{k\uparrow}\gamma_{k\uparrow} + v_k^* u_k \gamma_{k\downarrow}\gamma^\dagger_{k\downarrow} + u_k^2 \gamma^\dagger_{k\uparrow}\gamma^\dagger_{k\downarrow} - v_k^{*2} \gamma_{k\downarrow}\gamma_{k\uparrow}$$

$$c_{-k\downarrow} c_{k\uparrow} = -u_k^* v_k \gamma^\dagger_{k\uparrow}\gamma_{k\uparrow} + v_k u_k^* \gamma^\dagger_{k\downarrow}\gamma_{k\downarrow} + u^{*2} \gamma_{k\downarrow}\gamma_{k\uparrow} - v_k^2 \gamma^\dagger_{k\uparrow}\gamma^\dagger_{k\downarrow}$$

$$\Rightarrow \quad 2 \xi_k u_k v_k + \Delta_k^* v_k^2 - \Delta_k u_k^2 = 0 \quad \Big| \cdot \frac{\Delta_k^*}{u_k^2}$$

$$\Rightarrow \quad 2 \xi_k \Delta_k^* \frac{v_k}{u_k} + \frac{\Delta_k^{*2} v_k^2}{u_k^2} - |\Delta_k|^2 = 0$$

Quadratische Gl. für $\Delta_k^* \frac{v_k}{u_k}$

$$\Rightarrow \quad \frac{\Delta_k^* v_k}{u_k} = (\xi_k^2 + |\Delta_k|^2)^{1/2} - \xi_k = E_k - \xi_k$$

$$\Rightarrow \quad \cancel{\left|\frac{v_k}{u_k}\right| =} \quad \frac{|v_k|}{|u_k|} = \frac{E_k - \xi_k}{\Delta_k}$$

$$|u_k|^2 + |v_k|^2 = 1$$

$$\Rightarrow \quad |v_k|^2 = 1 - |u_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

Aufgabe 2 b)

$$\gamma_{k\uparrow} |\psi_G\rangle = \gamma_{-k\downarrow} |\psi_G\rangle = 0$$

$$|\psi_G\rangle = \prod_k \left(u_k + v_k c^+_{k\uparrow} c^+_{-k\downarrow} \right) |0\rangle$$

$$\gamma_{k\uparrow} |\psi_G\rangle = \left(u_k c_{k\uparrow} - v_k c^+_{-k\downarrow} \right) \prod_l \left(u_l + v_l c^+_{l\uparrow} c^+_{-l\downarrow} \right) |0\rangle$$

$$= \prod_{l \neq k} \left(u_l + v_l c^+_{l\uparrow} c^+_{-l\downarrow} \right) \Big[\underbrace{\cancel{u_k^2} c_{k\uparrow} + u_k v_k c_{k\uparrow} c^+_{k\uparrow} c^+_{-k\downarrow} - u_k v_k c^+_{-k\downarrow}}_{\text{heben sich weg}} - v_k^2 \cancel{c^+_{-k\downarrow} c^+_{k\uparrow} c^+_{-k\downarrow}} \Big] |0\rangle$$

$$= 0$$

particle number in ground state is not fixed

norm $\langle \psi_G | \psi_G \rangle = 1$, as $|u_k|^2 + |v_k|^2 = 1$

$$\bar{N} = \langle N \rangle = \langle \psi_G | \sum_k (c_{k\uparrow}^+ c_{k\uparrow} + c_{k\downarrow}^+ c_{k\downarrow}) | \psi_G \rangle$$

$$= 2 \sum_k \langle 0 | \prod_{e \neq k} (u_e^* + v_e^* c_{e\downarrow} c_{e\uparrow}) (u_k^* + v_k^* c_{-k\downarrow} c_{k\uparrow}) \, c_{k\uparrow}^+ c_{k\uparrow} \, (u_k + v_k c_{k\uparrow}^+ c_{-k\downarrow}^+) \prod_{e \neq k} (u_e + v_e c_{e\uparrow}^+ c_{-e\downarrow}^+) | 0 \rangle$$

$$= 2 \sum_k \langle 0 | \prod_{e \neq k} \left(|u_e|^2 + u_e^* v_e c_{e\uparrow}^+ c_{-e\downarrow}^+ + v_e^* u_e c_{-e\downarrow} c_{e\uparrow} + |v_e|^2 c_{-e\downarrow} c_{e\uparrow} c_{e\uparrow}^+ c_{-e\downarrow}^+ \right)$$

$$\left(|u_k|^2 c_{k\downarrow}^+ c_{k\downarrow} + u_k^* v_k c_{k\uparrow}^+ c_{k\uparrow} c_{k\uparrow}^+ c_{-k\downarrow}^+ + v_k^* u_k c_{-k\downarrow} c_{k\uparrow} c_{k\uparrow}^+ c_{k\uparrow} + |v_k|^2 c_{-k\downarrow} c_{k\uparrow} c_{k\uparrow}^+ c_{k\uparrow} c_{k\uparrow}^+ c_{-k\downarrow}^+ \right) | 0 \rangle$$

$\langle 0 | c_2 c_1 c_1^+ c_2^+ | 0 \rangle = -\langle 0 | c_2 \underbrace{c_1^+ c_1}_{=0} c_2^+ | 0 \rangle + \langle 0 | c_2 c_2^+ | 0 \rangle = 1$

$$= 2 \sum_k \prod_{e \neq k} (|u_e|^2 + |v_e|^2) \, |v_k|^2 = 2 \sum_k |v_k|^2 \approx \text{\# of electrons in occupied states}$$

$\approx E_F N(0)$

Similarly

$$\langle (N - \bar{N})^2 \rangle = 4 \sum_k \underbrace{|u_k|^2 |v_k|^2}_{\neq 0 \text{ for } |\xi_k| \text{ near } E_F \text{ in range } |\xi_k| \leq \Delta} \neq 0 \implies \text{particle \# is not sharp}$$

but $\dfrac{\sqrt{\langle (N - \bar{N})^2 \rangle}}{\bar{N}} = \dfrac{\sqrt{\pi \Delta N(0)}}{\bar{N}} \approx \sqrt{\dfrac{\Delta}{E_F}} \, \bar{N}^{-1/2} \ll 1$

$N(0) E_F \approx \bar{N} \approx 10^{23}$

For most problems this uncertainty does not matter

$|\psi_G\rangle$ depends on phase $|\psi_G\rangle = \prod_k (|u_k| + |v_k| e^{i\varphi} c_{k\uparrow}^+ c_{-k\downarrow}^+) |0\rangle$

We can construct ground states with fixed N

$$|\psi_N\rangle = \int_0^{2\pi} d\varphi \, e^{-iN\varphi/2} \prod_k (|u_k| + |v_k| e^{i\varphi} c_{k\uparrow}^+ c_{-k\downarrow}^+ | 0 \rangle$$

picks out states with $N/2$ pairs occupied

$\implies$ phase is uncertain

$\implies$ uncertainty relation $\Delta N \Delta \varphi \gtrsim 1$