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Institut für theoretische Festkörperphysik
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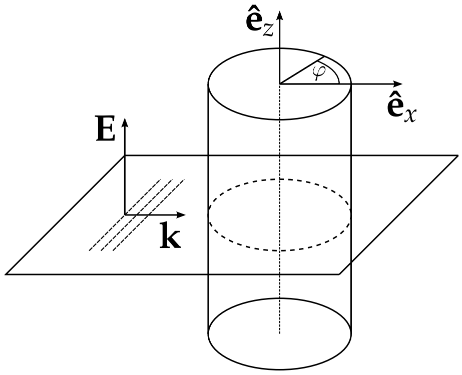
Problem set 8 for the course "Theoretical Optics" Sample Solutions

18 *All points will count as extra points.*

Diffraction at a Metallic Cylinder

We consider a monochromatic plane wave $\mathbf{E}_{\rm in}(x,t) = E_{\rm in}\hat{\mathbf{e}}_z e^{i(kx-\omega t)}$ that is diffracted at a perfectly conducting cylinder with radius a. The cylinder axis is oriented along the z-axis and infinitely extended along the z-direction (e.g., a very long, thin metallic wire). The symmetry of the problem favors a treatment in cylindrical coordinates, i.e., the electric field is generally written as $\mathbf{E}(\mathbf{r},t) = E_\rho\hat{\mathbf{e}}_\rho + E_\varphi\hat{\mathbf{e}}_\varphi + E_z\hat{\mathbf{e}}_z$ and the H-field accordingly. However, the diffracted field will keep its polarization state, so $\mathbf{E} \sim E_z(\rho,\varphi,z,t)\hat{\mathbf{e}}_z$. The total field is the sum of the incoming and scattered field: $\mathbf{E}_{\rm tot} = \mathbf{E}_{\rm in} + \mathbf{E}_{\rm sc}$.



a) Use Maxwell's equations in *cylindrical coordinates* to show that for z-polarized light

$$H_\rho = \sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{ik\rho}\frac{\partial E_z}{\partial\varphi} \qquad \text{and} \qquad H_\varphi = -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{ik}\frac{\partial E_z}{\partial\rho} \tag{1}$$

holds. [**2** Point(s)]

b) The z-component of the scattered field $E_{\rm sc} = E_{\rm tot} - E_{\rm in}$ obeys the wave equation in cylindrical coordinates

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2 + k^2\right)E_{\rm sc} = 0. \tag{2}$$

This equation can be decoupled by a separation ansatz $E_{\rm sc} = R(\rho)\Phi(\varphi)$, which yields two separate differential equations, one for $R(\rho)$ and one for $\Phi(\varphi)$. Those equations are coupled via a constant, which we denote by m^2. Argue, why this separation is allowed, state the two differential equations and find the general solution for $\Phi(\varphi)$. Further, show that m must be an integer. [**4** Point(s)]

c) Show that the differential equation for R can be recast into the form of the Bessel equation

$$x^2\frac{\partial^2}{\partial x^2}\tilde{R}(x) + x\frac{\partial}{\partial x}\tilde{R}(x) + \left(x^2 - m^2\right)\tilde{R}(x) = 0 \tag{3}$$

and give the correct expression for $\tilde{R}$ and x.[**2** Point(s)]

d) In the far field limit $k\rho \to \infty$, the scattered field must have the form of an outgoing cylindrical wave, i.e.,

$$E_{\text{sc}} \sim f(\varphi)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} \quad \text{for } k\rho \to \infty. \tag{4}$$

The outward propagating solutions to (**??**) satisfying the boundary condition (**??**) are given by the complex valued *Hankel functions of the first kind* $H_m^{(1)}$, which have the proper asymptotics far away from the cylinder:

$$H_m^{(1)}(k\rho) \simeq \sqrt{\frac{2}{\pi k\rho}}\mathrm{e}^{\mathrm{i}(k\rho - m\frac{\pi}{2} - \frac{\pi}{4})} \qquad \text{for } k\rho \gg 1. \tag{5}$$

The general outward propagating solution to (**??**) is given by

$$E_{\text{sc}}(\rho,\varphi) = E_{\text{in}} \sum_{m=-\infty}^{+\infty} A_m H_m^{(1)}(k\rho)\mathrm{e}^{\mathrm{i}m\varphi}, \qquad \text{with } A_m \in \mathbb{C}. \tag{6}$$

With the help of (**??**) and (**??**), find the expression for $f(\varphi)$ and show that

$$A_m = -\mathrm{i}^m \frac{J_m(ka)}{H_m^{(1)}(ka)}. \tag{7}$$

Here, J_m is a Bessel function of the first kind. [**4** Point(s)]

Hint: Be aware of the perfectly conducting boundary condition at the cylinder surface, i.e., $E_{\text{tot}}(\rho = a) = 0$. This gives you an equation for all the A_m, from which the coefficients can be projected out by multiplying with $\mathrm{e}^{-\mathrm{i}m'\varphi}$ and integrating from 0 to 2π. Make use of the formulae provided at the end of this problem set.

e) Show that in the far field limit the magnetic field $\mathbf{H}_{\text{sc}}$ only has a relevant transversal component H_φ and that the cycle-averged Poynting vector $\mathbf{S}_{\text{sc}}$ is consequently given by

$$\mathbf{S}_{\text{sc}} = -\frac{1}{2}\mathrm{Re}(E_{\text{sc},z}H^*_{\text{sc},\varphi})\hat{\mathbf{e}}_\rho. \quad \text{[5 Point(s)]} \tag{8}$$

Apply the asymptotic expansion *after* taking any derivatives! *Hint:* Show that H_ρ vanishes faster than H_φ for $k\rho \to \infty$. Make use of the formulae provided at the end of this problem set.

d) Now that we know the Poynting vector of the scattered field, we can compute the scattering cross section per height $\frac{\partial\sigma}{\partial z}$ given by

$$\frac{\partial\sigma}{\partial z} = \frac{1}{|\mathbf{S}_{\text{in}}|}\int_0^{2\pi} \mathbf{S}_{\text{sc}} \cdot \hat{\mathbf{e}}_\rho \rho \mathrm{d}\varphi. \tag{9}$$

This is a measure for the amount of the incident plane wave's power that is scattered into the outgoing cylindrical wave. Show that it has the value

$$\frac{4}{k}\sum_{m=-\infty}^{+\infty}\left|\frac{J_m(ka)}{H_m^{(1)}(ka)}\right|^2 \tag{10}$$

in the far field limit. [**3** Point(s)]

First we note the dispersion relation

$$\omega = \frac{k}{\sqrt{\mu_0 \mu \epsilon_0 \epsilon}}. \tag{11}$$

a) The curl-operator in cylindrical coordinates is given by (any standard text book on theoretical electrodynamics features this in an appendix or the first/last page of the hard cover, see, e.g. Griffiths, *Introduction to electrodynamics*)

$$\nabla \times \mathbf{E} = \left[\frac{1}{\rho}\frac{\partial}{\partial \varphi}E_z - \frac{\partial}{\partial z}E_\varphi\right]\hat{\mathbf{e}}_\rho + \left[\frac{\partial}{\partial z}E_\rho - \frac{\partial}{\partial \rho}E_z\right]\hat{\mathbf{e}}_\varphi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho E_\varphi) - \frac{\partial}{\partial \varphi}E_\rho\right]\hat{\mathbf{e}}_z. \tag{12}$$

Since only E_z is non-zero, the curl above simplifies to

$$\nabla \times \mathbf{E} = \frac{1}{\rho}\frac{\partial}{\partial \varphi}E_z\hat{\mathbf{e}}_\rho - \frac{\partial}{\partial \rho}E_z\hat{\mathbf{e}}_\varphi. \tag{13}$$

From Maxwell's curl equation

$$-\mu_0\mu\frac{\partial}{\partial t}\mathbf{H} = \nabla \times \mathbf{E} \tag{14}$$

we therefore find by inserting an appropriate plane wave ansatz ($\mathbf{H}(\mathbf{r},t) = \mathbf{H}_0 \mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$, we have seen this several times now) that

$$\begin{aligned} \mathrm{i}\omega\mu_0\mu\mathbf{H}(\mathbf{r},t) &= \nabla \times \mathbf{E}(\mathbf{r},t) && \text{(by (\textbf{??}))} && (15)\\ &= \frac{1}{\rho}\frac{\partial}{\partial \varphi}E_z\hat{\mathbf{e}}_\rho - \frac{\partial}{\partial \rho}E_z\hat{\mathbf{e}}_\varphi. && \text{(by (\textbf{??}))} && (16) \end{aligned}$$

Thus we have that

$$\begin{aligned} \mathbf{H}(\mathbf{r},t) &= \frac{1}{\mathrm{i}\omega\mu_0\mu}\left(\frac{1}{\rho}\frac{\partial}{\partial \varphi}E_z\hat{\mathbf{e}}_\rho - \frac{\partial}{\partial \rho}E_z\hat{\mathbf{e}}_\varphi\right) && (17)\\ &= H_\rho\hat{\mathbf{e}}_\rho - H_\varphi\hat{\mathbf{e}}_\varphi. && (18) \end{aligned}$$

Thus we can read off the components

$$\begin{aligned} H_\rho &= \frac{1}{\mathrm{i}\omega\mu_0\mu}\frac{1}{\rho}\frac{\partial}{\partial \varphi}E_z && && (19)\\ &= \sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{\mathrm{i}k\rho}\frac{\partial}{\partial \varphi}E_z, && \text{(by (\textbf{??}))} && (20)\\ H_\varphi &= -\frac{1}{i\omega\mu_0\mu}\frac{\partial}{\partial \rho}E_z && && (21)\\ &= -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{\mathrm{i}k}\frac{\partial}{\partial \rho}E_z. && \text{(by (\textbf{??}))} && (22) \end{aligned}$$

This is the desired result.

b) Inserting the separation ansatz $E_{\text{sc}} = R(\rho)\Phi(\varphi)$ into the wave equation (**??**) yields

$$R''\Phi + \frac{1}{\rho}R'\Phi + \frac{1}{\rho^2}R\Phi'' + k^2 R\Phi = 0, \tag{23}$$

where the prime $'$ denotes a derivative with respect to the argument. Multiplication with $\frac{\rho^2}{R\Phi}$ then results in

$$\rho^2 \frac{R''(\rho)}{R(\rho)} + \rho\frac{R'(\rho)}{R(\rho)} + \rho^2 k^2 = -\frac{\Phi''(\varphi)}{\Phi(\varphi)}. \tag{24}$$

The left hand side of this equation depends solely on the variable spatial ρ (k is a fixed parameter) and the right hand side depends only on the spatial variable φ. ρ and φ may take on all their respective allowed values indepently from one another, hereby inevitably changeing the values of the left hand side or right hand side. In order to be able to fulfill this equation for all ρ and φ then, both sides must be constant (independent of ρ and φ) and equal. As the problem text suggests, we denote this constant by m^2, and hence the two equations decouple and we have

$$\frac{\Phi''}{\Phi} = -m^2, \tag{25}$$

$$\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + \rho^2 k^2 = m^2. \tag{26}$$

For now, m is an unknown constant. Rewriting the first equation as

$$\Phi'' + m^2\Phi = 0, \tag{27}$$

we readily find the solution

$$\Phi(\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}. \tag{28}$$

Up to this point, m is still an arbitrary number.

Now φ is an angle, and a 360° rotation of the system around the z-axis creates the same pysical setup with the same solutions. This leads to the condition

$$\Phi(\varphi + 2\pi) \stackrel{!}{=} \Phi(\varphi). \tag{29}$$

Inserting this condition into the general solution (**??**), we find

$$\mathrm{e}^{\mathrm{i}m(\varphi+2\pi)} \stackrel{!}{=} \mathrm{e}^{\mathrm{i}m\varphi}, \tag{30}$$

and after dividing by $\mathrm{e}^{\mathrm{i}m\varphi}$ we end up with

$$\mathrm{e}^{\mathrm{i}m2\pi} \stackrel{!}{=} 1. \tag{31}$$

This equation can only be fulfilled, if $m \in \mathbb{Z}$. **Note that this is a typical quantization condition arising from physical boundary conditions of differential equations. Nothing 'quantum' has happened here, this is all classical physics.**

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c) Multiplying the second ODE (**??**) by R, we find

$$\rho^2 \frac{\partial^2}{\partial \rho^2} R(\rho) + \rho \frac{\partial}{\partial \rho} R(\rho) + \left(\rho^2 k^2 - m^2\right) R(\rho) = 0. \tag{32}$$

Now we introduce the function $\tilde{R}$ as

$$\begin{aligned} \tilde{R} \quad &: \quad x \mapsto \tilde{R}(x), &(33)\\ \tilde{R}(k\rho) \quad &:= \quad R(\rho), &(34) \end{aligned}$$

which defines the substitution of variable

$$x := k\rho. \tag{35}$$

Now we have to replace the derivatives of R by those of $\tilde{R}$:

$$\begin{aligned} (\frac{\partial}{\partial \rho})^n R(\rho) &= (\frac{\partial}{\partial \rho})^n \left[\tilde{R}(k\rho)\right] && \text{(by (??))} &(36)\\ &= (\frac{\partial}{\partial x})^n \tilde{R}(x)|_{x=k\rho} k^n && \text{(chain rule)} &(37) \end{aligned}$$

We substitute this result into (**??**) and obtain

$$(k\rho)^2 \frac{\partial^2}{\partial x^2} \tilde{R}(x)|_{x=k\rho} + (k\rho) \frac{\partial}{\partial x} \tilde{R}(x)|_{x=k\rho} + \left(\rho^2 k^2 - m^2\right) R(x)|_{x=k\rho} = 0. \tag{38}$$

Applying the subsitution (**??**) yields the desired Bessel equation for $\tilde{R}(x)$:

$$x^2 \frac{\partial^2}{\partial x^2} \tilde{R}(x) + x \frac{\partial}{\partial x} \tilde{R}(x) + \left(x^2 - m^2\right) \tilde{R}(x) = 0. \tag{39}$$

This was to show.

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d) In order to exploit the outgoing wave boundary condition (**??**), we have to look at the asymptotic form of the general solution for $k\rho \to \infty$, hence we substitute (**??**) into (**??**), which yields

$$\begin{aligned} E_{\text{sc}}(\rho, \varphi) &= E_{\text{in}} \sum_{m=-\infty}^{+\infty} A_m \sqrt{\frac{2}{\pi k \rho}} \mathrm{e}^{\mathrm{i}(k\rho - m\frac{\pi}{2} - \frac{\pi}{4})} \mathrm{e}^{\mathrm{i}m\varphi} && \text{(by (??)) and (??)} &(40)\\ &= \frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{\rho k}} \underbrace{E_{\text{in}} \sqrt{\frac{2}{\pi}} \sum_{m=-\infty}^{+\infty} A_m e^{\mathrm{i}(m\varphi - m\frac{\pi}{2} - \frac{\pi}{4})}}_{=:f(\varphi)}, && \text{(rearranged terms)} &(41) \end{aligned}$$

so we can already identify the function $f(\varphi)$ from (**??**) as

$$f(\varphi) = E_{\text{in}} \sqrt{\frac{2}{\pi}} \sum_{m=-\infty}^{+\infty} A_m e^{\mathrm{i}(m\varphi - m\frac{\pi}{2} - \frac{\pi}{4})}. \tag{42}$$

To find an expression for the A_m, we exploit the perfect electrical conductor (PEC) boundary condition on the surface of the cylinder ($\rho = a$), which yields

$$\mathbf{E}_{\text{tot}}(\rho = a, \varphi) = \mathbf{E}_{\text{in}}(\rho = a, \varphi) + \mathbf{E}_{\text{scat}}(\rho = a, \varphi) \stackrel{!}{=} 0. \tag{43}$$

The first equality stems from the continuity of the fields, the second from the PEC. We insert the incoming plane wave (as given in the problem text) and the form of the general solution of the scattered field (**??**). **Note that we are not in the asymptotic regime (far away from the cylinder) but in the vicinity of the cylinder. Therefore we have to use the full Hankel functions here.** Additionally, this equation is valid only for the cylinder surface $x = a\cos\varphi$, which we use to write (**??**) as

$$0 = E_{\text{in}} \left(\mathrm{e}^{\mathrm{i}kx} + \sum_{m=-\infty}^{+\infty} A_m H_m^{(1)}(ka) \mathrm{e}^{\mathrm{i}m\varphi} \right) \qquad ((\textbf{??}) \text{ with incoming plane wave and } (\textbf{??})) \tag{44}$$

$$= E_{\text{in}} \left(\mathrm{e}^{\mathrm{i}ka\cos\varphi} + \sum_{m=-\infty}^{+\infty} A_m H_m^{(1)}(ka) \mathrm{e}^{\mathrm{i}m\varphi} \right). \quad (x = a\cos\varphi) \tag{45}$$

The incoming amplitude E_{in} is irrelevant for the boundary condition and has been divided out of the equation in the following. As hinted, we project out the coefficients A_m by exploiting the orthogonality of the complex exponential functions as follows. We multiply (**??**) by $\mathrm{e}^{-\mathrm{i}m'\varphi}$ and integrate over φ to find

$$0 = \int_0^{2\pi} \left(\mathrm{e}^{\mathrm{i}ka\cos\varphi - m'\varphi} + \sum_{m=-\infty}^{+\infty} A_m H_m^{(1)}(ka) e^{\mathrm{i}(m-m')\varphi} \right) d\varphi \tag{46}$$

$$= \underbrace{\int_0^{2\pi} e^{\mathrm{i}ka\cos\varphi - m'\varphi} d\varphi}_{=\frac{2\pi}{(-i)^{m'}} J_{m'}(ka) \text{ by } (\textbf{??})} + \sum_{m=-\infty}^{+\infty} A_m H_m^{(1)}(ka) \underbrace{\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-m')\varphi} d\varphi}_{=2\pi\delta_{mm'} \text{ by } (\textbf{??})}. \quad \text{(integral is linear operator)} \tag{47}$$

Performing the sum and exploiting the Kroneckerdelta of (**??**), we end up with

$$0 = \frac{2\pi}{(-\mathrm{i})^m} J_m(ka) + 2\pi A_m H_m^{(1)}(ka), \tag{48}$$

which yields the solution

$$A_m = -\mathrm{i}^m \frac{J_m(ka)}{H_m^{(1)}(ka)} = \mathrm{i}^{m+2} \frac{J_m(ka)}{H_m^{(1)}(ka)}. \tag{49}$$

These are the desired coefficients.

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e) The cycle-averaged Poyting vector is given by $\mathbf{S} = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*)$. In cylindrical coordinates, we can write the cross product explicitly as

$$\mathbf{E} \times \mathbf{H}^* = \left(E_\varphi H_z^* - E_z H_\varphi^* \right) \hat{\mathbf{e}}_\rho + \left(E_z H_\rho^* - E_\rho H_z^* \right) \hat{\mathbf{e}}_\varphi + \left(E_\rho H_\varphi^* - E_\varphi H_\rho^* \right) \hat{\mathbf{e}}_z. \tag{50}$$

Since the electric field only has a z-component, we are left with two terms (analogous to part a):

$$\mathbf{E} \times \mathbf{H}^* = -E_z H_\varphi^* \hat{\mathbf{e}}_\rho + E_z H_\rho^* \hat{\mathbf{e}}_\varphi. \tag{51}$$

First, we consider the ρ-component of the scattered magnetic field H^{sc}_ρ, given by (**??**) of part a)

$$H^{\mathrm{sc}}_\rho = \sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{\mathrm{i}k\rho}\frac{\partial E^{\mathrm{sc}}_z}{\partial\varphi} \qquad \text{(by (\textbf{??}))} \tag{52}$$

$$= E_{\mathrm{in}}\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{\mathrm{i}k\rho}\frac{\partial}{\partial\varphi}\left(\sum_m A_m H^{(1)}_m(k\rho)\mathrm{e}^{\mathrm{i}m\varphi}\right) \qquad \text{(by (\textbf{??}))} \tag{53}$$

$$= E_{\mathrm{in}}\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{k\rho}\left(\sum_m mA_m H^{(1)}_m(k\rho)\mathrm{e}^{\mathrm{i}m\varphi}\right) \qquad \text{(performed derivative by } \varphi) \tag{54}$$

Inserting the asymptotic expansion (**??**) yields the field for $k\rho \gg 1$:

$$H^{\mathrm{sc}}_\rho \simeq E_{\mathrm{in}}\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}(k\rho)^{-\frac{3}{2}}\sqrt{\frac{2}{\pi}}\left(\sum_m mA_m \mathrm{e}^{\mathrm{i}\left(m\varphi - m\frac{\pi}{2} - \frac{\pi}{4}\right)}\right)\mathrm{e}^{\mathrm{i}k\rho}. \tag{55}$$

Similarly, we evaluate $H_{\mathrm{sc},\varphi}$

$$H^{\mathrm{sc}}_\varphi = -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{\mathrm{i}k}\frac{\partial E_{\mathrm{sc},z}}{\partial\rho} \qquad \text{(by (\textbf{??}))} \tag{56}$$

$$= -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{E_{\mathrm{in}}}{\mathrm{i}k}\left(\sum_m A_m\frac{\partial}{\partial\rho}H^{(1)}_m(k\rho)\mathrm{e}^{\mathrm{i}m\varphi}\right) \qquad \text{(by (\textbf{??}))} \tag{57}$$

$$= -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{E_{\mathrm{in}}}{\mathrm{i}}\left(\sum_m A_m\frac{\partial}{\partial k\rho}H^{(1)}_m(k\rho)\mathrm{e}^{\mathrm{i}m\varphi}\right) \qquad \text{(rearranged } k) \tag{58}$$

$$= -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{E_{\mathrm{in}}}{\mathrm{i}}\left(\sum_m A_m\frac{1}{2}\left(H^{(1)}_{m-1}(k\rho) - H^{(1)}_{m+1}(k\rho)\right)\mathrm{e}^{\mathrm{i}m\varphi}\right) \text{(applied (\textbf{??}))} \tag{59}$$

$$\simeq -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{E_{\mathrm{in}}}{\mathrm{i}}\left(\sum_m \frac{1}{2}A_m\sqrt{\frac{2}{\pi k\rho}}\underbrace{\left(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right)}_{=2\mathrm{i}}\mathrm{e}^{\mathrm{i}\left(m\varphi + k\rho - m\frac{\pi}{2} - \frac{\pi}{4}\right)}\right) \text{(applied (\textbf{??}))} \tag{60}$$

$$= -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}E_{\mathrm{in}}\sqrt{\frac{2}{\pi k\rho}}\left(\sum_m A_m\mathrm{e}^{\mathrm{i}\left(m\varphi - m\frac{\pi}{2} - \frac{\pi}{4}\right)}\right)\mathrm{e}^{\mathrm{i}k\rho}. \tag{61}$$

The problem text suggests to compare the magnitudes of the magnetic field components with each other, so we have a look at the ratio

$$\frac{H^{\mathrm{sc}}_\rho}{H^{\mathrm{sc}}_\varphi} = \frac{(k\rho)^{-\frac{3}{2}}}{(k\rho)^{-\frac{1}{2}}}\underbrace{\left(-\frac{\sum_m mA_m\mathrm{e}^{\mathrm{i}m\left(\varphi - \frac{\pi}{2}\right)}}{\sum_m A_m\mathrm{e}^{\mathrm{i}m\left(\varphi - \frac{\pi}{2}\right)}}\right)}_{=:\alpha} \tag{62}$$

$$= \frac{1}{k\rho}\alpha \to 0 \quad \text{for } k\rho \to \infty. \tag{63}$$

So since α is just some constant, the magnitude of the ρ-component of the magnetic field is much smaller than the magnitude of the φ-component, i.e., it is suppressed by a factor $\frac{1}{k\rho}$, and can be neglected in the far field limit.

Therefore, the Poynting vector contains only the $\hat{\mathbf{e}}_\rho$ part of (**??**), giving the final expression for the cycle averaged Poynting vector as

$$\mathbf{S}_{\text{sc}} = -\frac{1}{2}\text{Re}(E_{\text{sc},z} H^*_{\text{sc},\varphi})\hat{\mathbf{e}}_\rho. \tag{64}$$

This was to be shown.

Remark for teaching assistants: One can also take the ρ-derivative after the asymptotic expansion. This simplifies the calculations, therefore this method was explicitly excluded in the problem text.

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e) (Remark for teaching assistants: Since we deal with an idealized infinitely extended system here, the regular total scattering cross section σ would be infinitely large as well. That's why we deal with $\frac{\partial\sigma}{\partial z}$ here, which has the dimension of length (instead of area, as σ has) and is a finite quantity.)

To calculate the cross section, we need the cycle-averaged Poynting vector of the incoming and the scattered wave. The absolute value of the cycle-averaged Poyting vector of the incoming plane wave is well known and given by

$$|\mathbf{S}_{\text{in}}| = \frac{1}{2}\sqrt{\epsilon_0/\mu_0}|E_{\text{in}}|^2. \tag{65}$$

Combining (**??**) and (**??**), we we can express the product from (**??**) as

$$\begin{aligned} E_{\text{sc},z} H^*_{\text{sc},\varphi} &= \frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{\rho k}} E_{\text{in}} \sqrt{\frac{2}{\pi}} \sum_m A_m \mathrm{e}^{\mathrm{i}(m\varphi - m\frac{\pi}{2} - \frac{\pi}{4})} \\ &\quad \times \left(-\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}} E_{\text{in}} \sqrt{\frac{2}{\pi k\rho}} \left(\sum_m A_m \mathrm{e}^{\mathrm{i}\left(m\varphi - m\frac{\pi}{2} - \frac{\pi}{4}\right)} \right) \mathrm{e}^{\mathrm{i}k\rho} \right)^* \qquad (66) \\ &= -|E_{\text{in}}|^2 \frac{2}{\pi k\rho} \sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{m,m'} A_m A^*_{m'} \mathrm{e}^{\mathrm{i}(m-m')\varphi} \mathrm{e}^{\mathrm{i}(m-m')\frac{\pi}{2}} \qquad (67) \end{aligned}$$

Dividing by (**??**) and putting this into (**??**) yields (note the cancellation of the minus sign)

$$\begin{aligned} \frac{\partial\sigma}{\partial z} &= \frac{2}{\pi k} \sum_{m,m'} A_m A^*_{m'} \mathrm{e}^{\mathrm{i}(m-m')\frac{\pi}{2}} \underbrace{\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-m')\varphi} \mathrm{d}\varphi}_{=2\pi\delta_{mm'} \text{ by (??)}} \qquad (68) \\ &= \frac{4}{k} \sum_m |A_m|^2 \qquad \text{(evaluated sum and Kroneckerdelta)} \qquad (69) \\ &= \frac{4}{k} \sum_m \left| \frac{J_m(ka)}{H^{(1)}_m(ka)} \right|^2. \qquad (70) \end{aligned}$$

This was to show.

Useful formulae:

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-m')\phi}\mathrm{d}\phi = 2\pi\delta_{mm'} \quad \text{with} \quad \delta_{mm'} = \begin{cases} 1 & : m = m', \\ 0 & : \text{otherwise}, \end{cases} \tag{71}$$

$$J_n(x) = \frac{(-\mathrm{i})^n}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(x\cos\phi - n\phi)}\mathrm{d}\phi, \tag{72}$$

$$2\frac{\mathrm{d}}{\mathrm{d}x} H_m^{(1)}(x) = H_{m-1}^{(1)}(x) - H_{m+1}^{(1)}(x). \tag{73}$$

— Hand in solutions in lecture on 08.07.2012 —