Karlsruher Institut für Technologie (KIT) SS2012
Institut für theoretische Festkörperphysik
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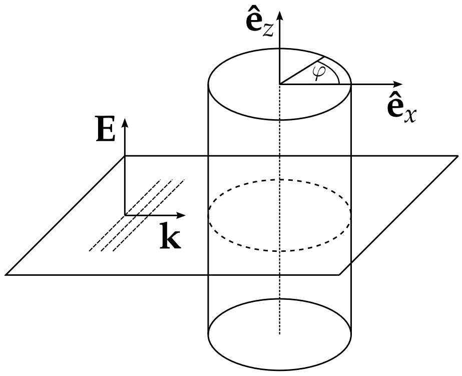
Problem set 8 for the course "Theoretical Optics"

18 *All points will count as extra points.*

Diffraction at a Metallic Cylinder



We consider a monochromatic plane wave $\mathbf{E}_{\text{in}}(x,t) = E_{\text{in}}\hat{\mathbf{e}}_z e^{i(kx-\omega t)}$ that is diffracted at a perfectly conducting cylinder with radius a. The cylinder axis is oriented along the z-axis and infinitely extended along the z-direction (e.g., a very long, thin metallic wire). The symmetry of the problem favors a treatment in cylindrical coordinates, i.e., the electric field is generally written as $\mathbf{E}(\mathbf{r},t) = E_\rho\hat{\mathbf{e}}_\rho + E_\varphi\hat{\mathbf{e}}_\varphi + E_z\hat{\mathbf{e}}_z$ and the H-field accordingly. However, the diffracted field will keep its polarization state, so $\mathbf{E} \sim E_z(\rho,\varphi,z,t)\hat{\mathbf{e}}_z$. The total field is the sum of the incoming and scattered field: $\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{in}} + \mathbf{E}_{\text{sc}}$.

a) Use Maxwell's equations in *cylindrical coordinates* to show that for z-polarized light

$$H_\rho = \sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{ik\rho}\frac{\partial E_z}{\partial\varphi} \qquad \text{and} \qquad H_\varphi = -\sqrt{\frac{\epsilon_0\epsilon}{\mu_0\mu}}\frac{1}{ik}\frac{\partial E_z}{\partial\rho} \tag{1}$$

holds. [**2** Point(s)]

b) The z-component of the scattered field $E_{\text{sc}} = E_{\text{tot}} - E_{\text{in}}$ obeys the wave equation in cylindrical coordinates

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2 + k^2\right)E_{\text{sc}} = 0. \tag{2}$$

This equation can be decoupled by a separation ansatz $E_{\text{sc}} = R(\rho)\Phi(\varphi)$, which yields two separate differential equations, one for $R(\rho)$ and one for $\Phi(\varphi)$. Those equations are coupled via a constant, which we denote by m^2. Argue, why this separation is allowed, state the two differential equations and find the general solution for $\Phi(\varphi)$. Further, show that m must be an integer. [**4** Point(s)]

c) Show that the differential equation for R can be recast into the form of the Bessel equation

$$x^2\frac{\partial^2}{\partial x^2}\tilde{R}(x) + x\frac{\partial}{\partial x}\tilde{R}(x) + \left(x^2 - m^2\right)\tilde{R}(x) = 0 \tag{3}$$

and give the correct expression for $\tilde{R}$ and x.[**2** Point(s)]

d) In the far field limit $k\rho \to \infty$, the scattered field must have the form of an outgoing cylindrical wave, i.e.,

$$E_{\mathrm{sc}} \sim f(\varphi)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} \quad \text{for } k\rho \to \infty. \tag{4}$$

The outward propagating solutions to (3) satisfying the boundary condition (4) are given by the complex valued *Hankel functions of the first kind* $H_m^{(1)}$, which have the proper asymptotics far away from the cylinder:

$$H_m^{(1)}(k\rho) \simeq \sqrt{\frac{2}{\pi k\rho}}\mathrm{e}^{\mathrm{i}(k\rho - m\frac{\pi}{2} - \frac{\pi}{4})} \qquad \text{for } k\rho \gg 1. \tag{5}$$

The general outward propagating solution to (2) is given by

$$E_{\mathrm{sc}}(\rho,\varphi) = E_{\mathrm{in}}\sum_{m=-\infty}^{+\infty} A_m H_m^{(1)}(k\rho)\mathrm{e}^{\mathrm{i}m\varphi}, \qquad \text{with } A_m \in \mathbb{C}. \tag{6}$$

With the help of (5) and (4), find the expression for $f(\varphi)$ and show that

$$A_m = -\mathrm{i}^m\frac{J_m(ka)}{H_m^{(1)}(ka)}. \tag{7}$$

Here, J_m is a Bessel function of the first kind. [**4** Point(s)]

Hint: Be aware of the perfectly conducting boundary condition at the cylinder surface, i.e., $E_{\mathrm{tot}}(\rho = a) = 0$. This gives you an equation for all the A_m, from which the coefficients can be projected out by multiplying with $\mathrm{e}^{-\mathrm{i}m'\varphi}$ and integrating from 0 to 2π. Make use of the formulae provided at the end of this problem set.

e) Show that in the far field limit the magnetic field $\mathbf{H}_{\mathrm{sc}}$ only has a relevant transversal component H_φ and that the cycle-averged Poynting vector $\mathbf{S}_{\mathrm{sc}}$ is consequently given by

$$\mathbf{S}_{\mathrm{sc}} = -\frac{1}{2}\mathrm{Re}(E_{\mathrm{sc},z}H^*_{\mathrm{sc},\varphi})\hat{\mathbf{e}}_\rho. \quad [\mathbf{5}\text{ Point(s)}] \tag{8}$$

Apply the asymptotic expansion *after* taking any derivatives! *Hint:* Show that H_ρ vanishes faster than H_φ for $k\rho \to \infty$. Make use of the formulae provided at the end of this problem set.

d) Now that we know the Poynting vector of the scattered field, we can compute the scattering cross section per height $\frac{\partial\sigma}{\partial z}$ given by

$$\frac{\partial\sigma}{\partial z} = \frac{1}{|\mathbf{S}_{\mathrm{in}}|}\int_0^{2\pi}\mathbf{S}_{\mathrm{sc}}\cdot\hat{\mathbf{e}}_\rho\rho\mathrm{d}\varphi. \tag{9}$$

This is a measure for the amount of the incident plane wave's power that is scattered into the outgoing cylindrical wave. Show that it has the value

$$\frac{4}{k}\sum_{m=-\infty}^{+\infty}\left|\frac{J_m(ka)}{H_m^{(1)}(ka)}\right|^2 \tag{10}$$

in the far field limit. [**3** Point(s)]

Useful formulae:

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-m')\phi}\mathrm{d}\phi = 2\pi\delta_{mm'} \quad \text{with} \quad \delta_{mm'} = \begin{cases} 1 & : m = m', \\ 0 & : \text{otherwise}, \end{cases} \tag{11}$$

$$J_n(x) = \frac{(-\mathrm{i})^n}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(x\cos\phi - n\phi)}\mathrm{d}\phi, \tag{12}$$

$$2\frac{\mathrm{d}}{\mathrm{d}x} H_m^{(1)}(x) = H_{m-1}^{(1)}(x) - H_{m+1}^{(1)}(x). \tag{13}$$

— Hand in solutions in lecture on 08.07.2012 —