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Tutorial:
☐ **Group 1,**
☐ **Group 2,**
☐ **Group 3.**

Name:____

Problem set 6 for the course "Theoretical Optics"
Sample Solutions

16 Operator Algebra In Quantum Mechanics

Note: All of these exercises can be done in a few lines.

We consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

along with the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2}$$

and a scalar product for vectors $u, v \in \mathbb{C}^2$ defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_i u_i^* v_i \in \mathbb{C}. \tag{3}$$

The commutator for two matrices A, B is defined as usual as

$$[A, B] := AB - BA, \tag{4}$$

where the standard matrix-matrix product is used.

a) Show that $[\sigma_1, \sigma_2] = 2\mathrm{i}\sigma_3$. [**1** Point(s)]

b) Show that for arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \in \mathbb{C}^2$ and complex numbers $\alpha, \beta \in \mathbb{C}$ the above scalar product is sesquilinear ("one-and-a-half times linear"), meaning

$$\langle \mathbf{c}, \alpha\mathbf{a} + \beta\mathbf{b} \rangle = \alpha\langle \mathbf{c}, \mathbf{a} \rangle + \beta\langle \mathbf{c}, \mathbf{b} \rangle \tag{5}$$

and

$$\langle \alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{c} \rangle = \alpha^*\langle \mathbf{a}, \mathbf{c} \rangle + \beta^*\langle \mathbf{b}, \mathbf{c} \rangle. \tag{6}$$

Furthermore, show that $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle^*$. [**1** Point(s)]

c) Find the eigenvalues and eigenvectors $\mathbf{v}_+$ and $\mathbf{v}_-$ of the matrix σ_2 such that $\langle \mathbf{v}_+, \mathbf{v}_+ \rangle = 1 = \langle \mathbf{v}_-, \mathbf{v}_- \rangle$. Express these eigenvectors as linear combinations of $\mathbf{v}_{1,2}$. Express $\mathbf{v}_{1,2}$ as linear combinations of $\mathbf{v}_\pm$. [**2** Point(s)]

d) Functions $f(M)$ of matrices M are evaluated by inserting the matrix into the Taylor series expansion of the desired function. Thus, the result is again a matrix and can be applied to a vector. Use this information to evaluate the matrix-vector products

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_{1,2} \tag{7}$$

in terms of $\mathbf{v}_\pm$ where θ is an arbitrary real angle. How does the use of the eigenvectors of σ_2 ease the computation? [**1** Point(s)]

Now we look at a linear operator $\hat{\sigma}_2$ in a two-dimensional complex linear space spanned by state vectors $|1\rangle, |2\rangle$. Any arbitrary vector $|\Psi\rangle$ in that space can be unambiguosly represented as a linear combination of these two states as $|\Psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle, \quad \alpha_{1,2} \in \mathbb{C}$.

The operator $\hat{\sigma}_2$ acts on the basis kets as

$$\hat{\sigma}_2|1\rangle = \mathrm{i}|2\rangle, \tag{8}$$

$$\hat{\sigma}_2|2\rangle = -\mathrm{i}|1\rangle. \tag{9}$$

For the basis states $|1\rangle$ and $|2\rangle$ (the kets) we define formally linear operators $\langle 1|$ and $\langle 2|$ (the bras) that map $|1\rangle$ and $|2\rangle$ to scalars (complex numbers). They are fully defined via their actions and their linearity ($\langle\cdot|\cdot\rangle$ is called bracket = bra ket):

$$\langle i|j\rangle := \langle i| \cdot |j\rangle := \delta_{ij} := \begin{cases} 1 & : i = j, \\ 0 & : \text{(otherwise)}, \end{cases} \tag{10}$$

$$\langle i| \cdot \big[\alpha|j\rangle + \beta|k\rangle\big] = \alpha\langle i|j\rangle + \beta\langle i|k\rangle. \qquad \text{(linearity)} \tag{11}$$

For any other states, such as the ket $|\Psi\rangle$ above, the corresponding bra is defined as (note the complex conjugation)

$$\langle\Psi| = \alpha_1^*\langle 1| + \alpha_2^*\langle 2|. \tag{12}$$

e) Only by using the algebraic properties given above, find for $|\Psi\rangle$ as given above the expectation value

$$\langle\Psi|\hat{\sigma}_2|\Psi\rangle := \langle\Psi| \cdot \big[\hat{\sigma}_2|\Psi\rangle\big]. \tag{13}$$

[**1** Point(s)]

f) Only by using the algebraic properties given above, find for given $|\Psi\rangle$ the expansion coefficients α_1 and α_2 with the help of $\langle 1|$ and $\langle 2|$. [**1** Point(s)]

g) Only by using the algebraic properties given above, show that the states

$$|+\rangle := \frac{1}{N_+}(|1\rangle + \mathrm{i}|2\rangle), \tag{14}$$

$$|-\rangle := \frac{1}{N_-}(|1\rangle - \mathrm{i}|2\rangle) \tag{15}$$

are eigenstates of the operator $\hat{\sigma}_2$ and find the eigenvalues. Determine the normalization constants $N_\pm \in \mathbb{C}$ such that $\langle +|+\rangle = 1 = \langle -|-\rangle$. Furthermore, show that $\langle +|-\rangle = 0$. Express $|1\rangle$ and $|2\rangle$ as linear combinations of $|+\rangle$ and $|-\rangle$. [**2** Point(s)]

h) Functions f of linear operators are evaluated by inserting the operator into the Taylor series expansion of the desired function. The result is again a linear operator and can act on a state. Only by using the algebraic properties given above, evaluate

$$\exp(\mathrm{i}\hat{\sigma}_2\theta) \cdot |i\rangle, \quad i = 1, 2, \tag{16}$$

in terms of $|+\rangle$ and $|-\rangle$ where θ is an arbitrary real angle (operators commute with numbers). How does the use of the eigenvectors of $\hat{\sigma}_2$ ease the computation? [1 Point(s)]

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a) We insert the matrices into the commutator

$$[\sigma_1, \sigma_2] = \sigma_1 \cdot \sigma_2 - \sigma_2 \cdot \sigma_1 \tag{17}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} +\mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} - \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & +\mathrm{i} \end{pmatrix} \tag{19}$$

$$= 2\mathrm{i} \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{20}$$

$$= 2\mathrm{i}\sigma_3. \tag{21}$$

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b) We insert the vectors intor the definition of the scalar product:

$$\langle \mathbf{c}, \alpha\mathbf{a} + \beta\mathbf{b} \rangle = \sum_i c_i^*(\alpha a_i + \beta b_i) \tag{22}$$

$$= \alpha \sum_i c_i^* a_i + \beta \sum_i c_i^* b_i \tag{23}$$

$$= \alpha\langle \mathbf{c}, \mathbf{a} \rangle + \beta\langle \mathbf{c}, \mathbf{b} \rangle. \tag{24}$$

Same goes for the other equation to show:

$$\langle \alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{c} \rangle = \sum_i (\alpha a_i + \beta b_i)^* c_i \tag{25}$$

$$= \alpha^* \sum_i a_i^* c_i + \beta^* \sum_i b_i^* c_i \tag{26}$$

$$= \alpha^*\langle \mathbf{a}, \mathbf{c} \rangle + \beta^*\langle \mathbf{b}, \mathbf{c} \rangle. \tag{27}$$

Finally we show

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_i a_i^* b_i \tag{28}$$

$$= \sum_i (a_i b_i^*)^* \tag{29}$$

$$= (\sum_i a_i b_i^*)^* \tag{30}$$

$$= (\sum_i b_i^* a_i)^* \tag{31}$$

$$= \langle \mathbf{b}, \mathbf{a} \rangle^*. \tag{32}$$

c) We determine the eigenvalues by solving for the zeros of the characteristic polynomial:

$$\det(\sigma_2 - \lambda \mathbb{1}) = \det \begin{pmatrix} -\lambda & -\mathrm{i} \\ \mathrm{i} & \lambda \end{pmatrix} \tag{33}$$

$$= \lambda^2 - 1 \overset{!}{=} 0. \tag{34}$$

The solutions to this equation and thus, the eigenvalues, are $\lambda_\pm = \pm 1$.

Eigenvectors can be determined as follows: let $\begin{pmatrix} a \\ b \end{pmatrix}$ be the eigenvector for eigenvalue $+1$, then we can deduce the equation

$$\sigma_2 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} -\mathrm{i}b \\ \mathrm{i}a \end{pmatrix} \tag{36}$$

$$\overset{!}{=} +1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}. \tag{37}$$

Arbitrarily choosing $a = 1$ determines $b = +\mathrm{i}$. This vector is not yet normalized, so we plugging this into the scalar product with unknown real normalization N:

$$\langle N \cdot \begin{pmatrix} 1 \\ +\mathrm{i} \end{pmatrix}, N \cdot \begin{pmatrix} 1 \\ +\mathrm{i} \end{pmatrix} \rangle = N^2 [\cdot 1 + (+\mathrm{i})^* \cdot (+\mathrm{i})] \tag{38}$$

$$= N^2 [1 + \underbrace{(-\mathrm{i}) \cdot (+\mathrm{i})}_{=+1}] \tag{39}$$

$$= N^2 [1 + 1] \tag{40}$$

$$= N^2 2 \overset{!}{=} 1. \tag{41}$$

Hence, $N = \frac{1}{\sqrt{2}}$ is the proper normalization and we find the eigenvector $\mathbf{v}_+$ for eigenvalue $+1$ to be

$$\mathbf{v}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +\mathrm{i} \end{pmatrix} \tag{42}$$

$$= \frac{1}{\sqrt{2}} (\mathbf{v}_1 + \mathrm{i} \mathbf{v}_2), \tag{43}$$

$$\mathbf{v}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \tag{44}$$

$$= \frac{1}{\sqrt{2}} (\mathbf{v}_1 - \mathrm{i} \mathbf{v}_2), \tag{45}$$

The normalized eigenvector $\mathbf{v}_-$ for eigenvalue -1 is found in the same fashion.

By inverting these relations we find the decomposition of $\mathbf{v}_{1,2}$ simply to be

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} (\mathbf{v}_+ + \mathbf{v}_-), \tag{46}$$

$$\mathbf{v}_2 = \frac{1}{\mathrm{i}\sqrt{2}} (\mathbf{v}_+ - \mathbf{v}_-). \tag{47}$$

d) The series expansion of the single most important function in physics (the exponential function) is given by

$$\exp(x) := \sum_{m=0}^{\infty} \frac{1}{m!} x^m \tag{48}$$

$$= \sum_m c_m x^m. \qquad \text{(short form)} \tag{49}$$

Inserting the matrix σ_2 yields

$$\exp(\mathrm{i}\sigma_2\theta) = \sum_m c_m (\mathrm{i}\theta)^m \sigma_2^m. \tag{50}$$

We apply this to $\mathbf{v}_1$ in the form of the expansion into eigenvectors $\mathbf{v}_\pm$ as follows:

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_{1,2} = \frac{1}{\sqrt{2}} \sum_m c_m (\mathrm{i}\theta)^m \sigma_2^m (\mathbf{v}_+ + \mathbf{v}_-) \tag{51}$$

$$\tag{52}$$

Applying the matrix repeatedly to its eigenvectors yields particularly simple results:

$$\sigma_2^m \mathbf{v}_+ = (+1)^m \mathbf{v}_+, \tag{53}$$

$$\sigma_2^m \mathbf{v}_- = (-1)^m \mathbf{v}_-. \tag{54}$$

Plugging this into above equation yields

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_1 = \frac{1}{\sqrt{2}} \Big[\underbrace{\sum_m c_m (\mathrm{i}\theta)^m (+1)^m}_{\exp(\mathrm{i}\theta)} \mathbf{v}_+ + \underbrace{\sum_m c_m (\mathrm{i}\theta)^m (-1)^m}_{\exp(-\mathrm{i}\theta)} \mathbf{v}_- \Big] \tag{55}$$

$$= \frac{1}{\sqrt{2}} [\mathrm{e}^{\mathrm{i}\theta} \mathbf{v}_+ + \mathrm{e}^{-\mathrm{i}\theta} \mathbf{v}_-], \tag{56}$$

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_2 = \frac{1}{\mathrm{i}\sqrt{2}} [\mathrm{e}^{\mathrm{i}\theta} \mathbf{v}_+ - \mathrm{e}^{-\mathrm{i}\theta} \mathbf{v}_-]. \tag{57}$$

The evaluation for $\mathbf{v}_2$ is analogous.

In principle, we are done. However, we can also express the solution in the former basis of the vectors $\mathbf{v}_i$ by substituting $\mathbf{v}_\pm$ in above solutions:

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_1 = \frac{1}{\sqrt{2}} [\mathrm{e}^{\mathrm{i}\theta} \mathbf{v}_+ + \mathrm{e}^{-\mathrm{i}\theta} \mathbf{v}_-] \tag{58}$$

$$= \frac{1}{2} [\mathrm{e}^{\mathrm{i}\theta} (\mathbf{v}_1 + \mathrm{i}\mathbf{v}_2) + \mathrm{e}^{-\mathrm{i}\theta} (\mathbf{v}_1 - \mathrm{i}\mathbf{v}_2)] \tag{59}$$

$$= \frac{1}{2} [(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \mathbf{v}_1 + \mathrm{i}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) \mathbf{v}_2] \tag{60}$$

$$= \cos(\theta) \mathbf{v}_1 - \sin(\theta) \mathbf{v}_2 \tag{61}$$

$$= \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}, \tag{62}$$

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_2 = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}. \tag{63}$$

The evaluation for $\mathbf{v}_2$ is analogous. While we are at it, from this form we readily read of the matrix $\exp(\mathrm{i}\sigma_2\theta)$ (was not asked for in the problem text):

$$\exp(\mathrm{i}\sigma_2\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}. \tag{64}$$

This is a rotation of $-\theta$ around the z-axis in the xy-plane. (Remark for teaching assistants: Yeah, I should have used $\exp(-\mathrm{i}\sigma_2\theta)$, stupid mistake...)

The use of eigenvectors lets us replace the complicated matrix σ_2 by a regular number (the eigenvalue) inside the argument of the function: $\exp(\mathrm{i}\sigma_2\theta)\mathbf{v}_\pm = \exp(\mathrm{i}(\pm1)\theta)\mathbf{v}_\pm$, which eases the evaluation.

Now we basically do the same computations in Dirac notation. **In the first part, we defined a scalar product and showed its algebraic properties, which are those of a sesquilinearform. Here we defined the algebraic properties of a sesquilinearform and use it as a scalar product.**

e) We begin by evaluating

$$\begin{aligned}
\hat\sigma_2|\Psi\rangle &= \hat\sigma_2[\alpha_1|1\rangle + \alpha_2|2\rangle] && \text{(def. of arbitrary } |\Psi\rangle) && (65)\\
&= \alpha_1\hat\sigma_2|1\rangle + \alpha_2\hat\sigma_2|2\rangle && \text{(linear operator)} && (66)\\
&= \alpha_1\mathrm{i}|2\rangle - \alpha_2\mathrm{i}|1\rangle && \text{(definition of } \hat\sigma_2) && (67)\\
&= -\mathrm{i}(\alpha_2|1\rangle - \alpha_1|2\rangle). && && (68)
\end{aligned}$$

Now we apply $\langle\Psi|$ as given in the problem text to this state:

$$\begin{aligned}
\underbrace{[\alpha_1^*\langle 1| + \alpha_2^*\langle 2|.]}_{=\langle\Psi|} \cdot \underbrace{[-\mathrm{i}(\alpha_2|1\rangle - \alpha_1|2\rangle)]}_{=\hat\sigma_2|\Psi\rangle} &= -\mathrm{i}[\alpha_1^*\alpha_2\underbrace{\langle 1|1\rangle}_{=1} - \alpha_1^*\alpha_1\underbrace{\langle 1|2\rangle}_{=0} \\
&\quad - \alpha_2^*\alpha_2\underbrace{\langle 2|1\rangle}_{=0} - \alpha_2^*\alpha_1\underbrace{\langle 2|2\rangle}_{=1}] && (69)\\
&= -\mathrm{i}[\alpha_1^*\alpha_2 - \alpha_2^*\alpha_1] && (70)\\
&= +\mathrm{i}[\alpha_1\alpha_2^* - (\alpha_1\alpha_2^*)^*] && (71)\\
&= 2\mathrm{i}\mathrm{Im}[\alpha_1\alpha_2^*]. && (72)
\end{aligned}$$

The result itself has no deeper meaning, it is the usage of bras and kets that matters here.

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f) By projecting $\langle\Psi|$ the on $|1\rangle$ we find

$$\langle 1|\Psi\rangle = \alpha_1\underbrace{\langle 1|1\rangle}_{=1} + \alpha_2\underbrace{\langle 1|2\rangle}_{=0}. \tag{73}$$

We find α_2 in a similar fashion and thus

$$\alpha_i = \langle i|\Psi\rangle. \tag{74}$$

These computations can also be done with the scalar product from the first part of the exercise with orthogonal vectors $\mathbf{v}_{1,2}$ and an arbitrary linear combination $\boldsymbol{\Psi} := \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$. Then we would obtain the expansion coefficients with respect to the orthonormalized basis as $\alpha_i = \langle \mathbf{v}_i, \boldsymbol{\Psi}\rangle$. The algebra is the same in different notation.

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g) Applying the operator gives

$$\hat{\sigma}_2|+\rangle = \frac{1}{N_+}(\underbrace{\hat{\sigma}_2|1\rangle}_{=\mathrm{i}|2\rangle} + \mathrm{i}\,\underbrace{\hat{\sigma}_2|2\rangle}_{=-\mathrm{i}|1\rangle}) \qquad \text{(linearity)} \tag{75}$$

$$= \frac{1}{N_+}(\mathrm{i}|2\rangle + |1\rangle) \tag{76}$$

$$= +|+\rangle. \tag{77}$$

In the same fashion we find that $|-\rangle$ is eigenstate for eigenvalue -1.

The normalization is found as

$$\langle +|+\rangle = \frac{1}{N_+}\frac{1}{N_+}^{*}[\langle 1| - \mathrm{i}\langle 2|] \cdot [|1\rangle + \mathrm{i}|2\rangle] \tag{78}$$

$$= \frac{1}{|N_+|^2}[\langle 1|1\rangle + \mathrm{i}\langle 1|2\rangle - \mathrm{i}\langle 2|1\rangle + \langle 2|2\rangle \qquad = \frac{2}{|N_+|^2} \overset{!}{=} 1. \tag{79}$$

Thus we find $N_+ = \sqrt{2}$ to be a suitable choice. (Remark for teaching assistants: At this point one could give the additional information, that normalized eigenvectors in a real linear space are determined up to a factor of ± 1 and in a complex linear space up to a phase $\mathrm{e}^{\mathrm{i}\theta}$, thus we could multiply such a phase to our eigenvector and it would still be a valid solution. However, we use the most simple result of a real positive normalization constant). In the same fashion, we find $N_- = N_+ = \frac{1}{\sqrt{2}}$.

Now, we have to show that

$$\langle +|-\rangle = \frac{1}{N_+^*}\frac{1}{N_-}[\langle 1| - \mathrm{i}\langle 2|] \cdot [\langle 1| - \mathrm{i}\langle 2|] \tag{80}$$

$$= \frac{1}{N_+^*}\frac{1}{N_-}[\langle 1|1\rangle - \mathrm{i}\langle 1|2\rangle - \mathrm{i}\langle 2|1\rangle - \langle 2|2\rangle] \tag{81}$$

$$= \frac{1}{N_+^*}\frac{1}{N_-}[1-1] \tag{82}$$

$$= 0. \tag{83}$$

With this result, we also find

$$\langle -|+\rangle = [\underbrace{\langle +|-\rangle}_{=0}]^* \tag{84}$$

$$= 0. \tag{85}$$

Since $\langle +|$ and $\langle -|$ are two orthonormalized vectors in a two-dimensional linear space with scalar product (a Hilbert space), they form a basis of that space. Hence, the expansions of

$\langle i|$ in terms of $\langle +|$ and $\langle -|$ can be found via projections regarding to the solution of f):

$$\langle +|1\rangle = \frac{1}{\sqrt{2}}, \tag{86}$$
$$\langle -|1\rangle = \frac{1}{\sqrt{2}}, \tag{87}$$
$$\langle +|2\rangle = \frac{-\mathrm{i}}{\sqrt{2}}, \tag{88}$$
$$\langle -|2\rangle = \frac{+\mathrm{i}}{\sqrt{2}}, \tag{89}$$

and we find

$$\langle 1| = \frac{1}{\sqrt{2}}[\langle +| + \langle -|], \tag{90}$$
$$\langle 2| = \frac{-\mathrm{i}}{\sqrt{2}}[\langle +| - \langle -|] \tag{91}$$
$$= \frac{1}{\mathrm{i}\sqrt{2}}[\langle +| - \langle -|]. \tag{92}$$

These are the same results as found in c), derived with the help of projections onto basis vectors.

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h) This is exactly exercise d), with $\sigma_2 \mapsto \hat{\sigma}_2$, $\mathbf{v}_\pm \mapsto |\pm\rangle$, $\mathbf{v}_i \mapsto |i\rangle$.

The crucial point is the following: exponential functions of abstract operators are evaluated by applying it to states expanded in terms of eigenstates of that operator. Then the complicated operator may be substituted by the eigenvalues in the function argument and the resulting state can be evaluated easily. Hence from now on we are allowed to write

$$\exp(\mathrm{i}\hat{\sigma}_2\theta)|1\rangle = \frac{1}{\sqrt{2}}[\exp(\mathrm{i}\hat{\sigma}_2\theta)|+\rangle + \exp(\mathrm{i}\hat{\sigma}_2\theta)|-\rangle] \tag{93}$$
$$= \frac{1}{\sqrt{2}}[\exp(+\mathrm{i}\theta)|+\rangle + \exp(-\mathrm{i}\theta)|-\rangle] \qquad \text{(eigenstates of } \hat{\sigma}_2\text{)} \tag{94}$$
$$= \cos(\theta)|1\rangle - \sin(\theta)|2\rangle. \tag{95}$$

That's the whole magic of linear algebra in quantum mechanics.

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17 Single Mode Cavity

We consider a quantum cavity along z-direction with a single mode of frequency ω and wavenumber k. The electric field shall be polarized in the x-direction and be in the initial state

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + \mathrm{e}^{\mathrm{i}\phi}|n+1\rangle\right), \tag{96}$$

where ϕ is a given phase and $|n\rangle$ is the Fock state with n photons.

a) Use the time-evolution operator to calculate how this state evolves in time, i.e. find $|\Psi(t)\rangle$. [**3** Point(s)]

b) Using the solution from a) calculate the expectation value of the electric field $\langle \hat{E}_x \rangle := \langle \Psi(t)|\hat{E}_x|\Psi(t)\rangle$ as well as the variance $\langle \hat{E}_x^2 \rangle := \langle \Psi(t)|\hat{E}_x\hat{E}_x|\Psi(t)\rangle$.
Use these results to determine the standard deviation

$$\Delta E_x = \sqrt{\langle (\hat{E}_x - \langle \hat{E}_x \rangle)^2 \rangle} = \sqrt{\langle \hat{E}_x^2 \rangle - \langle \hat{E}_x \rangle^2}. \tag{97}$$

Show that they have the forms (You may assume $E_0 \in \mathbb{R}$ here)

$$\langle \hat{E}_x \rangle = E_0 \mathcal{N} \sin(kz)\cos(\omega t - \phi), \tag{98}$$

$$\Delta E_x = \mathcal{N} |E_0 \sin(kz)| \sqrt{2 - \cos^2(\omega t - \phi)} \tag{99}$$

and determine the constant $\mathcal{N} \in \mathbb{R}$. [**5** Point(s)]

c) Analogously to b), calculate the standard deviation Δn for the number operator $\hat{n}$. [**2** Point(s)]

d) Show that the uncertainties from b) and c) fulfill the following relation (as known from the lecture):

$$(\Delta n)(\Delta E_x) \geq \frac{1}{2} |E_0 \sin(kz)| \, |\langle \hat{a}^\dagger - \hat{a} \rangle|. \tag{100}$$

[**2** Point(s)]

To simplify notation in this and the following subexercises, we introduce a rescaled energy as

$$\tilde{E}_n = \frac{E_n}{\hbar} = \omega \left(n + \frac{1}{2}\right). \tag{101}$$

a) The time-evolution operator is given by $e^{-\frac{i}{\hbar}\hat{H}t}$, where $\hat{H}$ is the Hamilton operator $\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$. Thus, we need to evaluate

$$|\Psi(t)\rangle := e^{-\frac{i}{\hbar}\hat{H}t}|\Psi(t=0)\rangle \tag{102}$$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{e^{-\frac{i}{\hbar}\hat{H}t}|n\rangle}_{\text{term 1}} + e^{i\phi}\underbrace{e^{-\frac{i}{\hbar}\hat{H}t}|n+1\rangle}_{\text{term 2}}\right) \qquad \text{(by (96), linear operator)} \tag{103}$$

We evaluate term 1, term 2 is analogous.

Though the evaluation of functions of operators should have already been discussed at length in the lecture, it is quite a novel concept for students and should be again explained in the problem session if possible.

A function of an operator is defined via the series representation of that function, for the exponential function, the Hamiltonian and the given Fock state it is

$$e^{-\frac{i}{\hbar}\hat{H}t}|n\rangle := \sum_{m=0}^{\infty} \frac{(-\frac{i}{\hbar}\hat{H}t)^m}{m!}|n\rangle \qquad \text{(Def. of exp}(\cdot)) \tag{104}$$

$$= \sum_{m=0}^{\infty} \frac{(-\frac{i}{\hbar}t)^m}{m!} \underbrace{\hat{H}^m|n\rangle}_{\text{to evaluate}}. \qquad \text{(Op. } \hat{H} \text{ commutes with numbers } m, i, \hbar, t) \tag{105}$$

In order to compute the exponential, we need to evaluate the repeated application of the Hamiltonian on the Fock state $|n\rangle$:

$$\hat{H}|n\rangle = \hbar\omega\left(\underbrace{\hat{n}}_{\text{operator}} + \frac{1}{2}\right)|n\rangle \qquad \text{(Single mode Hamiltonian)} \tag{106}$$

$$= \hbar\omega\left(\underbrace{n}_{\text{number}} + \frac{1}{2}\right)|n\rangle \qquad \text{(eigenstate of number operator)} \tag{107}$$

$$= E_n|n\rangle. \qquad \text{(definition of (101))} \tag{108}$$

Since the Hamiltonian commutes with the numbers E_n, we immediately have

$$\hat{H}^m|n\rangle = E_n^m|n\rangle. \tag{109}$$

With this, we can write

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|n\rangle = \sum_{m=0}^{\infty}\frac{(-\frac{\mathrm{i}}{\hbar}t)^m}{m!}E_n^m|n\rangle \qquad \text{(by (105)), (109))} \tag{110}$$

$$= \sum_{m=0}^{\infty}\frac{(-\frac{\mathrm{i}}{\hbar}E_n t)^m}{m!}|n\rangle \qquad \text{(rearranged terms)} \tag{111}$$

$$= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t}|n\rangle. \qquad \text{(def. of }\exp(\cdot)\text{)} \tag{112}$$

In summary, if an operator function is applied to an eigenstate of the operator involved, one may substitute the operator by the particular eigenvalue in the function argument. This should be accompanied by a justification, e.g. "because state is eigenstate of operator". The detailed computation need not be repeated then.

Finally, we can solve this exercise in quite a few lines:

$$|\Psi(t)\rangle := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|\Psi(t=0)\rangle \tag{113}$$

$$= \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|n\rangle + \mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}|n+1\rangle\right) \qquad \text{(by (96), linear operator)} \tag{114}$$

$$= \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t}|n\rangle + \mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_{n+1}t}|n+1\rangle\right) \qquad \text{(by (112))} \tag{115}$$

---

b) **We need to note some preliminary results first.** We recall the action of the laddder operators on Fock states:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{116}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{117}$$

The electric field operator is given as

$$\hat{E}_x = E_0\sin(kz)\left(\hat{a}^\dagger + \hat{a}\right). \tag{118}$$

In the following, the abbreviations of pure phase factors

$$c_n := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_n t}, \tag{119}$$

$$c := \mathrm{e}^{\mathrm{i}\phi} \tag{120}$$

will be quite useful.

The energy difference between two neighboring Fock states is

$$E_{n+1} - E_n = \hbar\omega, \qquad \text{(by (101))} \tag{121}$$

which we use to evaluate the products

$$c_n^* c_{n+1} = \mathrm{e}^{+\frac{\mathrm{i}}{\hbar}E_n t}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_{n+1} t} = \mathrm{e}^{-\mathrm{i}\omega t}, \tag{122}$$

$$c_{n+1}^* c_n = \left(c_n^* c_{n+1}\right)^* = \mathrm{e}^{+\mathrm{i}\omega t}. \tag{123}$$

Now we can start to actually evaluate the expectation values. We first evaluate $\hat{a}|\Psi(t)\rangle$ and $\hat{a}^\dagger|\Psi(t)\rangle$:

$$\begin{aligned}\hat{a}|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(c_n\hat{a}|n\rangle + cc_{n+1}\hat{a}|n+1\rangle\right), \\ &= \frac{1}{\sqrt{2}}\left(c_n\sqrt{n}|n-1\rangle + cc_{n+1}\sqrt{n+1}|n\rangle\right),\end{aligned}$$

and

$$\begin{aligned}\hat{a}^\dagger|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(c_n\hat{a}^\dagger|n\rangle + cc_{n+1}\hat{a}^\dagger|n+1\rangle\right), \\ &= \frac{1}{\sqrt{2}}\left(c_n\sqrt{n+1}|n+1\rangle + cc_{n+1}\sqrt{n+2}|n+2\rangle\right).\end{aligned}$$

Together, we can evaluate

$$\begin{aligned}\hat{E}_x|\Psi(t)\rangle &= \frac{E_0}{\sqrt{2}}\sin(kz)\Big[c_n\sqrt{n}|n-1\rangle + cc_{n+1}\sqrt{n+1}|n\rangle \\ &\qquad + c_n\sqrt{n+1}|n+1\rangle + cc_{n+1}\sqrt{n+2}|n+2\rangle\Big] \qquad (124) \\ &= \frac{E_0}{\sqrt{2}}\sin(kz)\Big[c_n\left(\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle\right) \\ &\qquad + cc_{n+1}\left(\sqrt{n+1}|n\rangle + \sqrt{n+2}|n+2\rangle\right)\Big] \qquad (125)\end{aligned}$$

The expectation value projects out only the contributions of $\langle n|$ and $\langle n+1|$ and discards the rest:

$$\begin{aligned}
\langle\Psi(t)|\hat{E}_x|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left[c_n^*\langle n| + c^* c_{n+1}^*\langle n+1|\right]\hat{E}_x|\Psi(t)\rangle \\
&= \frac{E_0}{2}\sin(kz)\Bigg[c\,\underbrace{c_n^* c_{n+1}}_{=\mathrm{e}^{-\mathrm{i}\omega t}}\sqrt{n+1}\,\underbrace{\langle n|n\rangle}_{=1} \\
&\qquad + c^*\,\underbrace{c_{n+1}^* c_n}_{=\mathrm{e}^{+\mathrm{i}\omega t}}\sqrt{n+1}\,\underbrace{\langle n+1|n+1\rangle}_{=1}\Bigg] \\
&= \frac{E_0}{2}\sqrt{n+1}\sin(kz)\underbrace{\Bigg[\underbrace{c\mathrm{e}^{-\mathrm{i}\omega t}}_{=\mathrm{e}^{-\mathrm{i}(\omega t-\phi)}} + \underbrace{c^*\mathrm{e}^{\mathrm{i}\omega t}}_{=\mathrm{e}^{+\mathrm{i}(\omega t-\phi)}}\Bigg]}_{=2\cos(\cdot)} \\
&= E_0\sqrt{n+1}\sin(kz)\cos(\omega t-\phi).
\end{aligned}$$

For the variance we use

$$\begin{aligned}
\hat{E}_x\hat{E}_x|\Psi(t)\rangle &= \frac{E_0}{\sqrt{2}}\sin(kz)\hat{E}_x\Big[c_n\left(\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle\right) && \text{(by (125))} \\
&\quad + cc_{n+1}\left(\sqrt{n+1}|n\rangle + \sqrt{n+2}|n+2\rangle\right)\Big] \\
&= \frac{E_0^2}{\sqrt{2}}\sin^2(kz)\Big[c_n\left(\sqrt{n}\hat{a}^\dagger|n-1\rangle + \sqrt{n+1}\hat{a}^\dagger|n+1\rangle\right) \\
&\quad + c_n\left(\sqrt{n}\hat{a}|n-1\rangle + \sqrt{n+1}\hat{a}|n+1\rangle\right) \\
&\quad + cc_{n+1}\left(\sqrt{n+1}\hat{a}^\dagger|n\rangle + \sqrt{n+2}\hat{a}^\dagger|n+2\rangle\right) \\
&\quad + cc_{n+1}\left(\sqrt{n+1}\hat{a}|n\rangle\sqrt{n+2}\hat{a}|n+2\rangle\right)\Big] && \text{(by (118))} \\
&= \frac{E_0^2}{\sqrt{2}}\sin^2(kz)\Big[c_n\left(\sqrt{n}\sqrt{n}|n\rangle + \sqrt{n+1}\sqrt{n+2}|n+2\rangle\right) \\
&\quad + c_n\left(\sqrt{n}\sqrt{n-1}|n-2\rangle + \sqrt{n+1}\sqrt{n+1}|n\rangle\right) \\
&\quad + cc_{n+1}\left(\sqrt{n+1}\sqrt{n+1}|n+1\rangle + \sqrt{n+2}\sqrt{n+3}|n+3\rangle\right) \\
&\quad + cc_{n+1}\left(\sqrt{n+1}\sqrt{n}|n-1\rangle + \sqrt{n+2}\sqrt{n+2}|n+1\rangle\right)\Big] && \text{(by (116))}
\end{aligned}$$

Similarly to the expectation value of $\hat{E}_x$ we find for the variance

$$
\begin{aligned}
\langle\Psi(t)|\hat{E}_x\hat{E}_x|\Psi(t)\rangle &= \frac{E_0^2}{2}\sin^2(kz)\left[c_n^*\langle n| + c^*c_{n+1}^*\langle n+1|\right] \\
&\quad\times\Big[c_n\left(\sqrt{n}\sqrt{n}|n\rangle + \sqrt{n+1}\sqrt{n+2}|n+2\rangle\right) \\
&\quad + c_n\left(\sqrt{n}\sqrt{n-1}|n-2\rangle + \sqrt{n+1}\sqrt{n+1}|n\rangle\right) \\
&\quad + cc_{n+1}\left(\sqrt{n+1}\sqrt{n+1}|n+1\rangle + \sqrt{n+2}\sqrt{n+3}|n+3\rangle\right) \\
&\quad + cc_{n+1}\left(\sqrt{n+1}\sqrt{n}|n-1\rangle + \sqrt{n+2}\sqrt{n+2}|n+1\rangle\right)\Big] \\
&= \frac{E_0^2}{2}\sin^2(kz)\Bigg[n\underbrace{\langle n|n\rangle}_{=1} + (n+1)\underbrace{\langle n|n\rangle}_{=1} \\
&\quad + (n+1)\underbrace{\langle n+1|n+1\rangle}_{=1} + (n+2)\underbrace{\langle n+1|n+1\rangle}_{=1}\Bigg] \\
&= \frac{E_0^2}{2}\sin^2(kz)\,(4n+4) \\
&= 2(n+1)E_0^2\sin^2(kz).
\end{aligned}
$$

For the standard deviation we therefore find

$$
\begin{aligned}
\Delta E_x &= \sqrt{\langle\hat{E}_x^2\rangle - \langle\hat{E}_x\rangle^2} \\
&= \sqrt{2(n+1)E_0^2\sin^2(kz) - \left[E_0^2(n+1)\sin^2(kz)\cos^2(\omega t-\phi)\right]} \qquad \text{(entered results)} \\
&= \sqrt{n+1}|E_0\sin(kz)|\sqrt{2-\cos^2(\omega t-\phi)}
\end{aligned}
$$

These are the forms that were wanted with $\mathcal{N} = \sqrt{n+1}$.

---

c) As above, we evalute

$$\hat{n}|\Psi(t)\rangle = \hat{n}\frac{1}{\sqrt{2}}\left(c_n|n\rangle + cc_{n+1}|n+1\rangle\right) \tag{126}$$

$$= \frac{1}{\sqrt{2}}\left(c_n n|n\rangle + cc_{n+1}(n+1)|n+1\rangle\right) \tag{127}$$

and

$$\hat{n}\hat{n}|\Psi(t)\rangle = \hat{n}\hat{n}\frac{1}{\sqrt{2}}\left(c_n|n\rangle + cc_{n+1}|n+1\rangle\right) \tag{128}$$

$$= \frac{1}{\sqrt{2}}\left(c_n n^2|n\rangle + cc_{n+1}(n+1)^2|n+1\rangle\right). \tag{129}$$

Recalling that $\langle n|n\rangle = 1 = \langle n+1|n+1\rangle$ and phases c, c_n, c_{n+1} all cancel to 1 in the expectation value due to the use of complex conjugates, we find

$$\langle\hat{n}\rangle = \frac{1}{2}\left(n + (n+1)\right) = n + \frac{1}{2} \tag{130}$$

and

$$\langle \hat{n}^2 \rangle = \frac{1}{2}\left(n^2 + (n+1)^2\right) = n^2 + n + \frac{1}{2}. \tag{131}$$

Plugging these results into the defintion of the spread, we find

$$\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2} \tag{132}$$

$$= \sqrt{\frac{1}{2}\left(n^2 + \underbrace{(n+1)^2}_{=n^2+2n+1}\right) - \underbrace{(n+\frac{1}{2})^2}_{=n^2+n+\frac{1}{4}}} \tag{133}$$

$$= \sqrt{n^2 + n + \frac{1}{2} - n^2 - n - \frac{1}{4}} \tag{134}$$

$$= \sqrt{\frac{1}{4}} = \frac{1}{2}. \tag{135}$$

---

d) In order to check the inequality, we need to evaluate $|\langle \hat{a}^\dagger - \hat{a} \rangle|$. So, using the results from b), we find

$$\begin{aligned}
\left(\hat{a}^\dagger - \hat{a}\right)|\Psi(t)\rangle =& \frac{1}{\sqrt{2}}\Big[c_n\sqrt{n+1}|n+1\rangle + c c_{n+1}\sqrt{n+2}|n+2\rangle \\
& -c_n\sqrt{n}|n-1\rangle - c c_{n+1}\sqrt{n+1}|n\rangle\Big] \\
=& \frac{1}{\sqrt{2}}\Big[c_n\left(\sqrt{n+1}|n+1\rangle - \sqrt{n}|n-1\rangle\right) \\
& + c c_{n+1}\left(\sqrt{n+2}|n+2\rangle - \sqrt{n+1}|n\rangle\right)\Big],
\end{aligned}$$

so

$$\begin{aligned}
\langle \Psi(t)|\left(\hat{a}^\dagger - \hat{a}\right)|\Psi(t)\rangle =& \frac{1}{2}\sqrt{n+1}\underbrace{\Big[\mathrm{e}^{\mathrm{i}\omega t}\underbrace{\mathrm{e}^{-\mathrm{i}\phi}}_{c^*}\underbrace{\langle n+1|n+1\rangle}_{=1} - \mathrm{e}^{-\mathrm{i}\omega t}\underbrace{\mathrm{e}^{\mathrm{i}\phi}}_{=c}\underbrace{\langle n|n\rangle}_{=1}\Big]}_{=2\mathrm{i}\sin(\cdot)} \\
=& \mathrm{i}\sqrt{n+1}\sin(\omega t - \phi).
\end{aligned}$$

With the solutions from the previous subexercises, the left hand side of the inequality is given by

$$(\Delta n)(\Delta E_x) = \frac{1}{2}|E_0 \sin(kz)|\underbrace{\sqrt{n+1}\sqrt{2 - \cos^2(\omega t - \phi)}}_{\overset{?}{\geq}|\langle \hat{a}^\dagger - \hat{a}\rangle|}. \tag{136}$$

Thus, we conclude that the inequality to show holds true as long as

$$\sqrt{2 - \cos^2(\omega t - \phi)} \geq |\sin(\omega t - \phi)|, \tag{137}$$

which can be rewritten as (but the argument is also already applicable to the form above)

$$\sqrt{1 + \sin^2(\omega t - \phi)} \geq |\sin(\omega t - \phi)|. \tag{138}$$

This inequality is certainly fulfilled, since the left-hand side yields values in $[1,2]$ while the right-hand side can only have values in $[0,1]$.

— Hand in solutions in lecture on —