Karlsruher Institut für Technologie (KIT) SS2012
Institut für theoretische Festkörperphysik
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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

☐ **Group 4.**

Name:______________________________

Problem set 7 for the course "Theoretical Optics"

16 Operator Algebra In Quantum Mechanics

Note: All of these exercises can be done in a few lines.

We consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

along with the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2}$$

and a scalar product for vectors $u, v \in \mathbb{C}^2$ defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_i u_i^* v_i \in \mathbb{C}. \tag{3}$$

The commutator for two matrices A, B is defined as usual as

$$[A, B] := AB - BA, \tag{4}$$

where the standard matrix-matrix product is used.

a) Show that $[\sigma_1, \sigma_2] = 2\mathrm{i}\sigma_3$. [1 Point(s)]

b) Show that for arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \in \mathbb{C}^2$ and complex numbers $\alpha, \beta \in \mathbb{C}$ the above scalar product is sesquilinear ("one-and-a-half times linear"), meaning

$$\langle \mathbf{c}, \alpha\mathbf{a} + \beta\mathbf{b} \rangle = \alpha \langle \mathbf{c}, \mathbf{a} \rangle + \beta \langle \mathbf{c}, \mathbf{b} \rangle \tag{5}$$

and

$$\langle \alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{c} \rangle = \alpha^* \langle \mathbf{a}, \mathbf{c} \rangle + \beta^* \langle \mathbf{b}, \mathbf{c} \rangle. \tag{6}$$

Furthermore, show that $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle^*$. [1 Point(s)]

c) Find the eigenvalues and eigenvectors $\mathbf{v}_+$ and $\mathbf{v}_-$ of the matrix σ_2 such that $\langle \mathbf{v}_+, \mathbf{v}_+ \rangle = 1 = \langle \mathbf{v}_-, \mathbf{v}_- \rangle$. Express these eigenvectors as linear combinations of $\mathbf{v}_{1,2}$. Express $\mathbf{v}_{1,2}$ as linear combinations of $\mathbf{v}_\pm$. [**2** Point(s)]

d) Functions $f(M)$ of matrices M are evaluated by inserting the matrix into the Taylor series expansion of the desired function. Thus, the result is again a matrix and can be applied to a vector. Use this information to evaluate the matrix-vector products

$$\exp(\mathrm{i}\sigma_2\theta) \cdot \mathbf{v}_{1,2} \tag{7}$$

in terms of $\mathbf{v}_\pm$ where θ is an arbitrary real angle. How does the use of the eigenvectors of σ_2 ease the computation? [**1** Point(s)]

Now we look at a linear operator $\hat{\sigma}_2$ in a two-dimensional complex linear space spanned by state vectors $|1\rangle, |2\rangle$. Any arbitrary vector $|\Psi\rangle$ in that space can be unambiguosly represented as a linear combination of these two states as $|\Psi\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle, \quad \alpha_{1,2} \in \mathbb{C}$.

The operator $\hat{\sigma}_2$ acts on the basis kets as

$$\hat{\sigma}_2|1\rangle = \mathrm{i}|2\rangle, \tag{8}$$

$$\hat{\sigma}_2|2\rangle = -\mathrm{i}|1\rangle. \tag{9}$$

For the basis states $|1\rangle$ and $|2\rangle$ (the kets) we define formally linear operators $\langle 1|$ and $\langle 2|$ (the bras) that map $|1\rangle$ and $|2\rangle$ to scalars (complex numbers). They are fully defined via their actions and their linearity ($\langle\cdot|\cdot\rangle$ is called bracket = bra ket):

$$\langle i|j\rangle := \langle i|\cdot|j\rangle := \delta_{ij} := \begin{cases} 1 & : i = j, \\ 0 & : \text{(otherwise)}, \end{cases} \tag{10}$$

$$\langle i| \cdot \big[\alpha|j\rangle + \beta|k\rangle\big] = \alpha\langle i|j\rangle + \beta\langle i|k\rangle. \qquad \text{(linearity)} \tag{11}$$

For any other states, such as the ket $|\Psi\rangle$ above, the corresponding bra is defined as (note the complex conjugation)

$$\langle\Psi| = \alpha_1^*\langle 1| + \alpha_2^*\langle 2|. \tag{12}$$

e) Only by using the algebraic properties given above, find for $|\Psi\rangle$ as given above the expectation value

$$\langle\Psi|\hat{\sigma}_2|\Psi\rangle := \langle\Psi| \cdot \big[\hat{\sigma}_2|\Psi\rangle\big]. \tag{13}$$

[**1** Point(s)]

f) Only by using the algebraic properties given above, find for given $|\Psi\rangle$ the expansion coefficients α_1 and α_2 with the help of $\langle 1|$ and $\langle 2|$. [**1** Point(s)]

g) Only by using the algebraic properties given above, show that the states

$$|+\rangle := \frac{1}{N_+}(|1\rangle + \mathrm{i}|2\rangle), \tag{14}$$

$$|-\rangle := \frac{1}{N_-}(|1\rangle - \mathrm{i}|2\rangle) \tag{15}$$

are eigenstates of the operator $\hat{\sigma}_2$ and find the eigenvalues. Determine the normalization constants $N_\pm \in \mathbb{C}$ such that $\langle +|+\rangle = 1 = \langle -|-\rangle$. Furthermore, show that $\langle +|-\rangle = 0$. Express $|1\rangle$ and $|2\rangle$ as linear combinations of $|+\rangle$ and $|-\rangle$. [**2** Point(s)]

h) Functions f of linear operators are evaluated by inserting the operator into the Taylor series expansion of the desired function. The result is again a linear operator and can act on a state. Only by using the algebraic properties given above, evaluate

$$\exp(\mathrm{i}\hat{\sigma}_2\theta) \cdot |i\rangle, \quad i = 1, 2, \tag{16}$$

in terms of $|+\rangle$ and $|-\rangle$ where θ is an arbitrary real angle (operators commute with numbers). How does the use of the eigenvectors of $\hat{\sigma}_2$ ease the computation? [**1** Point(s)]

17 Single Mode Cavity

We consider a quantum cavity along z-direction with a single mode of frequency ω and wavenumber k. The electric field shall be polarized in the x-direction and be in the initial state

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + \mathrm{e}^{\mathrm{i}\phi}|n+1\rangle\right), \tag{17}$$

where ϕ is a given phase and $|n\rangle$ is the Fock state with n photons.

a) Use the time-evolution operator to calculate how this state evolves in time, i.e. find $|\Psi(t)\rangle$. [**3** Point(s)]

b) Using the solution from a) calculate the expectation value of the electric field $\langle \hat{E}_x \rangle := \langle \Psi(t)|\hat{E}_x|\Psi(t)\rangle$ as well as the variance $\langle \hat{E}_x^2 \rangle := \langle \Psi(t)|\hat{E}_x\hat{E}_x|\Psi(t)\rangle$.
Use these results to determine the standard deviation

$$\Delta E_x = \sqrt{\langle (\hat{E}_x - \langle \hat{E}_x \rangle)^2 \rangle} = \sqrt{\langle \hat{E}_x^2 \rangle - \langle \hat{E}_x \rangle^2}. \tag{18}$$

Show that they have the forms (You may assume $E_0 \in \mathbb{R}$ here)

$$\langle \hat{E}_x \rangle = E_0 \mathcal{N} \sin(kz)\cos(\omega t - \phi), \tag{19}$$

$$\Delta E_x = \mathcal{N} |E_0 \sin(kz)| \sqrt{2 - \cos^2(\omega t - \phi)} \tag{20}$$

and determine the constant $\mathcal{N} \in \mathbb{R}$. [**5** Point(s)]

c) Analogously to b), calculate the standard deviation Δn for the number operator $\hat{n}$. [**2** Point(s)]

d) Show that the uncertainties from b) and c) fulfill the following relation (as known from the lecture):

$$(\Delta n)(\Delta E_x) \geq \frac{1}{2}|E_0 \sin(kz)|\,|\langle \hat{a}^\dagger - \hat{a} \rangle|. \tag{21}$$

[**2** Point(s)]

— Hand in solutions in tutorial on 02.07.2012 —