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Institut für theoretische Festkörperphysik
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**Tutorial:**

☐ **Group 1,**

☐ **Group 2,**

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**Name:**________________________________________

# Problem set 4 for the course "Theoretical Optics" Sample Solutions

## 7 Light Propagation in Anisotropic Media

The dispersion relation in a general anisotropic medium is given by the Fresnel equation

$$\left(\frac{c^2k^2}{\omega^2}\right)\left(\frac{\epsilon_x c^2 k_x^2}{\omega^2}+\frac{\epsilon_y c^2 k_y^2}{\omega^2}+\frac{\epsilon_z c^2 k_z^2}{\omega^2}\right) - \left(\frac{c^2k_x^2}{\omega^2}\epsilon_x\left(\epsilon_y+\epsilon_z\right)+\frac{c^2k_y^2}{\omega^2}\epsilon_y\left(\epsilon_x+\epsilon_z\right)+\frac{c^2k_z^2}{\omega^2}\epsilon_z\left(\epsilon_x+\epsilon_y\right)\right) + \epsilon_x\epsilon_y\epsilon_z = 0. \quad (1)$$

**a)** Show that the Fresnel equation can be recast into the form

$$s_x^2\left(v_p^2-v_y^2\right)\left(v_p^2-v_z^2\right)+s_y^2\left(v_p^2-v_x^2\right)\left(v_p^2-v_z^2\right)+s_z^2\left(v_p^2-v_x^2\right)\left(v_p^2-v_y^2\right)=0. \quad (2)$$

Here, we introduced the phase velocity $v_p = \frac{\omega}{|\mathbf{k}|}$ of the wave, the phase velocities along the coordinate axes $v_i = \frac{c}{\sqrt{\epsilon_i}}$ and the normalized wave vector components $s_i = \frac{k_i}{|\mathbf{k}|}$.
[ **4** Point(s) ]

**b)** Now, we want to demonstrate that, in general, there are two solutions (phase velocities) for every given propagation direction $\mathbf{s} = (s_x, s_y, s_z)$. To find these solutions, assume that $\epsilon_x < \epsilon_y < \epsilon_z$ and insert

$$v_x^2 = v_y^2 + q_x, \quad v_z^2 = v_y^2 - q_z, \quad v_p^2 = v_y^2 + q \quad (3)$$

into the Fresnel equation (2). Calculate the two possible solutions $q', q''$ for $q$ and show that they must have opposite sign, i.e., $q' \cdot q'' \leq 0$. Here, we choose $q' \geq 0$. Show that then, the following inequality holds:

$$-q_z \leq q'' \leq 0 \leq q' \leq q_x. \quad (4)$$

[ **4** Point(s) ]

**c)** There are two distinct directions, where only one solution exists, so $q' = q''$. These directions form the two optical axes of the crystal. Show that these optical axes must lie in the $(x,z)$-plane and demonstrate that the angle $\beta$ which the $z$-axis encloses with the two axes is given by

$$\tan\beta = \pm\sqrt{\frac{v_x^2 - v_y^2}{v_y^2 - v_z^2}}. \tag{5}$$

[ **3** Point(s) ]

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**a)** Multiplying the Fresnel equation with $\frac{\omega^4}{c^6 k^4}$ yields

$$\left(\frac{\epsilon_x}{c^2}\frac{k_x^2}{k^2} + \frac{\epsilon_y}{c^2}\frac{k_y^2}{k^2} + \frac{\epsilon_z}{c^2}\frac{k_z^2}{k^2}\right) - \left(\frac{\epsilon_x}{c^2}\left(\frac{\epsilon_y}{c^2}+\frac{\epsilon_z}{c^2}\right)\frac{k_x^2}{k^2} + \frac{\epsilon_y}{c^2}\left(\frac{\epsilon_x}{c^2}+\frac{\epsilon_z}{c^2}\right)\frac{k_y^2}{k^2} + \frac{\epsilon_z}{c^2}\left(\frac{\epsilon_x}{c^2}+\frac{\epsilon_y}{c^2}\right)\frac{k_z^2}{k^2}\right)\frac{\omega^2}{k^2} + \frac{\epsilon_x}{c^2}\frac{\epsilon_y}{c^2}\frac{\epsilon_z}{c^2}\frac{\omega^4}{k^4} = 0.$$

Then, inserting $\frac{\epsilon_i}{c^2} = \frac{1}{v_i^2}$, $s_i = \frac{k_i}{k}$ and $v_p = \frac{\omega}{k}$ leads to

$$\left(\frac{s_x^2}{v_x^2} + \frac{s_y^2}{v_y^2} + \frac{s_z^2}{v_z^2}\right) - \left(\frac{s_x^2}{v_x^2}\left(\frac{1}{v_y^2}+\frac{1}{v_z^2}\right) + \frac{s_y^2}{v_y^2}\left(\frac{1}{v_x^2}+\frac{1}{v_z^2}\right) + \frac{s_z^2}{v_z^2}\left(\frac{1}{v_x^2}+\frac{1}{v_y^2}\right)\right)v_p^2 + \frac{v_p^4}{v_x^2 v_y^2 v_z^2} = 0.$$

Further, multiplication with $v_x^2 v_y^2 v_z^2$ yields

$$\left(s_x^2 v_y^2 v_z^2 + s_y^2 v_x^2 v_z^2 + s_z^2 v_x^2 v_y^2\right) - v_p^2 s_x^2\left(v_y^2 + v_z^2\right) - v_p^2 s_y^2\left(v_x^2 + v_z^2\right) - v_p^2 s_z^2\left(v_x^2 + v_y^2\right) + 1\cdot v_p^4 = 0.$$

Exploiting the fact that $s_x^2 + s_y^2 + s_z^2 = 1$, we insert this factor in front of the last term $v_p^4$ and sort the terms according to $s_i$:

$$s_x^2\left(v_p^4 - v_p^2 v_y^2 - v_p^2 v_z^2 + v_y^2 v_z^2\right) + s_y^2\left(v_p^4 - v_p^2 v_x^2 - v_p^2 v_z^2 + v_x^2 v_z^2\right) + s_z^2\left(v_p^4 - v_p^2 v_x^2 - v_p^2 v_y^2 + v_x^2 v_y^2\right) = 0,$$

which is readily rewritten (just factor the binomials) in the desired form of (2)

$$s_x^2\left(v_p^2 - v_y^2\right)\left(v_p^2 - v_z^2\right) + s_y^2\left(v_p^2 - v_x^2\right)\left(v_p^2 - v_z^2\right) + s_z^2\left(v_p^2 - v_x^2\right)\left(v_p^2 - v_y^2\right) = 0. \tag{6}$$

[ **4** Point(s) ]

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**b)** From $\epsilon_x < \epsilon_y < \epsilon_z$ we know that $v_x > v_y > v_z$. Thus, both $q_x$ and $q_z$ must be positive. We need to express the parentheses of (2) in terms of $q$ and $q_i$. Therefore, subtracting the substitutions (3) given in the problem text from each other yields the expressions

$$v_p^2 - v_x^2 = q - q_x, \quad v_p^2 - v_y^2 = q, \quad v_p^2 - v_z^2 = q + q_z.$$

Inserting these expressions into Eq. (2) yields

$$s_x^2 q\left(q+q_z\right)+s_y^2\left(q-q_x\right)\left(q+q_z\right)+s_z^2 q\left(q-q_x\right)=0. \tag{7}$$

Collecting the powers of $q$, we find

$$q^2 \underbrace{\left(s_x^2+s_y^2+s_z^2\right)}_{=1}+q \underbrace{\left(s_x^2 q_z+s_y^2\left(q_z-q_x\right)-s_z^2 q_x\right)}_{=:b} \underbrace{-s_y^2 q_x q_z}_{=:c}=0. \tag{8}$$

The two solutions for $q$ are therefore given by

$$q^{\prime}=\frac{-b+\sqrt{b^2-4 c}}{2} \quad \text{and} \quad q^{\prime\prime}=\frac{-b-\sqrt{b^2-4 c}}{2}. \tag{9}$$

Forming the product, we find

$$q^{\prime} \cdot q^{\prime\prime}=\left(\frac{-b+\sqrt{b^2-4 c}}{2}\right)\left(\frac{-b-\sqrt{b^2-4 c}}{2}\right)=\frac{b^2}{4}-\frac{b^4-4 c}{4}=c. \tag{10}$$

Since both $q_x$ and $q_z$ are positive, $c=-s_y^2 q_x q_z$ must be negative. Thus, $q^{\prime} \cdot q^{\prime\prime} \leq 0$. Further, if $q>q_x$ or if $q<-q_z$, then all terms on the left-hand side of Eq. (7) would be positive. Thus, a solution would not be possible!

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**c)** Since $q^{\prime} \cdot q^{\prime\prime} \leq 0$, $q^{\prime}=q^{\prime\prime}$ can only be fulfilled if $q^{\prime}=q^{\prime\prime}=0$. Since $q^{\prime} \cdot q^{\prime\prime}=c$, we know that $c=0$. Consequently, in (9) both equations have to be fulfilled *simultaneously* (ruling out $b=\pm 1$), so we find that $b=0$ as well. Explicitly, we have the conditions

$$\begin{aligned} c &= -s_y^2 q_x q_z=0 \\ b &= s_x^2 q_z+s_y^2\left(q_z-q_x\right)-s_z^2 q_x=0 \end{aligned}$$

Since we only consider the truely biaxial case ($\epsilon_x<\epsilon_y<\epsilon_z$), $q_x$ and $q_z$ can not be zero. Thus,

$$s_y=0 \quad \text{and} \quad s_x^2 q_z+s_y^2\left(q_z-q_x\right)-s_z^2 q_x=0. \tag{11}$$

Inserting the first condition into the second, we end up with

$$s_y=0 \quad \text{and} \quad s_x^2 q_z=s_z^2 q_x. \tag{12}$$

Thus, the axes must lie in the $(x,z)$-plane and the angle $\beta$ with the $z$-axis can be introduced via $s_x=\sin\beta$ and $s_y=\cos\beta$. Solving for $\beta$ then results in

$$\frac{s_x^2}{s_z^2}=\tan(\beta)^2=\frac{q_x}{q_z}=\frac{v_x^2-v_y^2}{v_y^2-v_z^2} \quad \rightarrow \quad \tan\beta=\sqrt{\frac{v_x^2-v_y^2}{v_y^2-v_z^2}}. \tag{13}$$

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## 8 Reflection at Faraday Rotators

We consider the reflection of a normally incident linearly polarized plane wave

$$\mathbf{E}_{\mathrm{i}}(z, t)=E_{\mathrm{i}} \hat{\mathbf{e}}_x e^{\mathrm{i}(k z-\omega t)} \tag{14}$$

at an air-material interface. We want to determine the polarization and intensity of the reflected wave.

Inside the material, the transmitted wave propagates as two circularly polarized plane waves with refractive indices $n_+$ and $n_-$ as

$$\mathbf{E}_\mathrm{t} = E_{\mathrm{t},+} \begin{pmatrix} 1 \\ +\mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}(k_+ z - \omega t)} + E_{\mathrm{t},-} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}(k_- z - \omega t)}, \tag{15}$$

with the two dispersion relations

$$k_\pm = n_\pm \omega / c. \tag{16}$$

For the electric field amplitudes we assume $E_\mathrm{i}, E_{\mathrm{t},\pm} \in \mathbb{C}$.

**a)** Consider the incoming linearly polarized wave as a superposition of circularly polarized waves (compare Prob. 5). Show, that the incoming circular parts can only excite transmitted/reflected circular parts of the same polarization orientation. [**1** Point(s)]

**b)** Use the result from a) to determine the amplitude of the reflected waves in terms of $n_\pm$ for each polarization state independently. *Hint:* Use the continuity conditions for the $E$- and $H$-field components to derive a set of equations relating $E_\mathrm{t}$ and $E_\mathrm{r}$ to $E_\mathrm{i}$ (transmitted, incoming and reflected amplitude), incorporating the respective indices of refraction. Be aware of phase jumps and use $\mu = \mu_0$ (a commonly used approximation for optical frequencies). [**1.5** Point(s)]

**c)** Show that the full reflected wave fulfills

$$\frac{|\mathbf{E}_\mathrm{r}|^2}{|\mathbf{E}_\mathrm{i}|^2} = \frac{1}{2}\left[\left(\frac{1-n_+}{1+n_+}\right)^2 + \left(\frac{1-n_-}{1+n_-}\right)^2\right]. \tag{17}$$

[**1** Point(s)]

**d)** What is the polarization state of the reflected wave? [**0.5** Point(s)]

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**a)** (REMARK FOR TEACHING ASSISTANTS: The derivation here follows the standard treatment of normally incident light (see, e.g., Griffiths, *Introduction to electrodynamics*), albeit spiced up by the two polarization states, see Fig. 1).

Defining the complex polarization vectors

$$\mathbf{p}_\pm = \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix}, \tag{18}$$

we can write $\hat{\mathbf{e}}_x = \frac{1}{2}(\mathbf{p}_+ + \mathbf{p}_-)$ and the incoming plane wave as

$$E_\mathrm{i}\hat{\mathbf{e}}_x e^{\mathrm{i}(kz-\omega t)} = (\underbrace{E_\mathrm{i}\frac{1}{2}}_{=:E_{\mathrm{i},+}}\mathbf{p}_+ + \underbrace{E_\mathrm{i}\frac{1}{2}}_{=:E_{\mathrm{i},-}}\mathbf{p}_-)\mathrm{e}^{\mathrm{i}(kz-\omega t)}, \tag{19}$$

the superposition of two circularly polarized plane waves with opposite orientation and half the amplitudes of the linearly polarized wave.

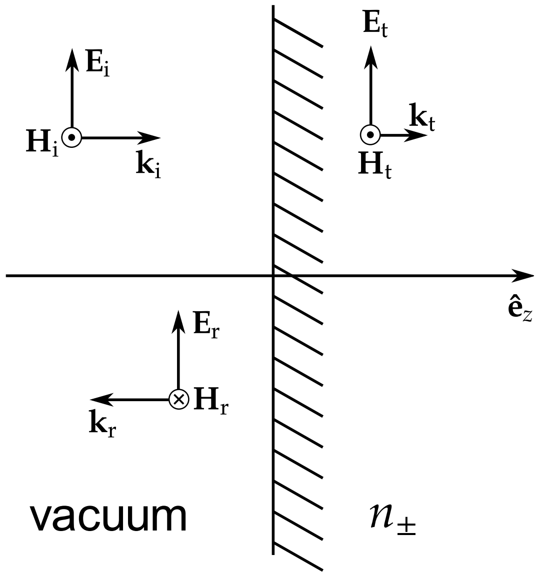


Figure 1: The fields and wave vectors involved in the reflection process.

At the interface $z = 0$, the total parallel electric field components must be continuous. Due to the normal incidence, we already only deal with parallel components. Using this continuity condition in combination with (15) and (19) at $z = 0$ yields

$$(E_{\mathrm{i},+}\mathbf{p}_+ + E_{\mathrm{i},-}\mathbf{p}_-)\mathrm{e}^{-\mathrm{i}\omega t} + \mathbf{E}_\mathrm{r}\mathrm{e}^{\mathrm{i}(\mathbf{k}_\mathrm{r}\mathbf{r}_\parallel - \omega t)} \stackrel{!}{=} (E_{\mathrm{t},+}\mathbf{p}_+ + E_{\mathrm{t},-}\mathbf{p}_-)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{20}$$

This condition must hold at all points $\mathbf{r}_\parallel = (x, y, 0)$ on the interface, thus the reflected wave vector $\mathbf{k}_\mathrm{r}$ can only have a $z$-component $\mathbf{k}_\mathrm{r} = k_\mathrm{r}\hat{\mathbf{e}}_z$ (that does not occur in the above equation). Dividing by the common exponential factor $\mathrm{e}^{-\mathrm{i}\omega t}$ we end up with

$$(E_{\mathrm{i},+}\mathbf{p}_+ + E_{\mathrm{i},-}\mathbf{p}_-) + \mathbf{E}_\mathrm{r} - (E_{\mathrm{t},+}\mathbf{p}_+ + E_{\mathrm{t},-}\mathbf{p}_-) = 0, \tag{21}$$

where $\mathbf{E}_\mathrm{r}$ is the unknown complex valued amplitude of the reflected plane wave.

(Remark for teaching assistants: Most of the arguments leading to this expression should already be known to the students. This is just an elaborate derivation here. However, they should mention that the continuity conditions lead to this last expression.)

We note that the complex polarization vectors are orthogonal

$$\mathbf{p}_\pm^* \cdot \mathbf{p}_\pm = \begin{pmatrix} 1 \\ \mp\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix} \tag{22}$$

$$= 1 + (\mp\mathrm{i})(\pm\mathrm{i}) \tag{23}$$

$$= 2, \tag{24}$$

$$\mathbf{p}_\mp^* \cdot \mathbf{p}_\pm = \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix} \tag{25}$$

$$= 1 + (\pm\mathrm{i})(\pm\mathrm{i}) \tag{26}$$

$$= 0. \tag{27}$$

Now we can project (21) onto the vectors $\mathbf{p}_\pm$, i.e., we multiply from the left with $\mathbf{p}_\pm^*$

(the complex conjugate). We start with the + part:

$$\begin{aligned}
0 &= \mathbf{p}_+^* \cdot (E_{\mathrm{i},+}\mathbf{p}_+ + E_{\mathrm{i},-}\mathbf{p}_- + \mathbf{E}_\mathrm{r}) - \mathbf{p}_+^* \cdot (E_{\mathrm{t},+}\mathbf{p}_+ + E_{\mathrm{t},-}\mathbf{p}_-) && (28)\\
&= (E_{\mathrm{i},+} \underbrace{\mathbf{p}_+^* \cdot \mathbf{p}_+}_{=2} + E_{\mathrm{i},-} \underbrace{\mathbf{p}_+^* \cdot \mathbf{p}_-}_{=0} + \mathbf{p}_+^* \cdot \mathbf{E}_\mathrm{r}) - (E_{\mathrm{t},+} \underbrace{\mathbf{p}_+^* \cdot \mathbf{p}_+}_{=2} + E_{\mathrm{t},-} \underbrace{\mathbf{p}_+^* \cdot \mathbf{p}_-}_{=0}) && (29)\\
&= 2E_{\mathrm{i},+} + \mathbf{p}_+^* \cdot \mathbf{E}_\mathrm{r} - 2E_{\mathrm{t},+}. && (30)
\end{aligned}$$

Defining

$$E_{\mathrm{r},\pm} := \frac{1}{2}\mathbf{p}_\pm^* \cdot \mathbf{E}_\mathrm{r}, \qquad (31)$$

we end up with two distinct continuity equations for the two orthogonal circular polarizations:

$$E_{\mathrm{i},\pm} + E_{\mathrm{r},\pm} = E_{\mathrm{t},\pm}. \qquad (32)$$

Thus, each of the two polarization states can be investigated separately.

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**b)** (Remark for teaching assistants: Again, the derivation here follows the standard treatment of normally incident light (see, e.g., Griffiths, *Introduction to electrodynamics*). Since we reduced the problem in part a) to the standard case, one can exactly follow the text book treatment here.)

We can look at one polarization first and thus can drop the ± subscript. The electric field amplitudes involved by (32) then obey

$$E_\mathrm{i} + E_\mathrm{r} = E_\mathrm{t}. \qquad (33)$$

This is one equation for the two unknowns $E_\mathrm{r}$ and $E_\mathrm{t}$. Hence, we need a second condition to solve this system of linear equations. This condition comes from exploiting the continuity of the parallel component of the magetic field $\mathbf{H}$, which reads

$$\mathbf{H}_{\mathrm{i},\parallel} + \mathbf{H}_{\mathrm{r},\parallel} = \mathbf{H}_{\mathrm{t},\parallel}. \qquad (34)$$

We can look at the polarizations independently, so we start with the magnetic field associated with the $\mathbf{p}_+$ state and use Maxwell's equation

$$-\frac{\partial}{\partial t}\mathbf{B} = \nabla \times \mathbf{E}, \qquad (35)$$

with $\mathbf{E}(r,t) = E\mathbf{p}_+ \mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$ and $\mathbf{B} = \mu_0\mathbf{H}$. Substituting this into Maxwell's equation above yields

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{H} &= -\frac{E}{\mu_0}\nabla \times (\mathbf{p}_+ \mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}) && \text{(amplitude } E \text{ is constant)} && (36)\\
&= -\mathrm{i}\frac{E}{\mu_0}(\mathbf{k} \times \mathbf{p}_+)\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)} && \text{(performed curl)} && (37)\\
\Rightarrow \mathbf{H} &= \frac{E}{\omega\mu_0}(\mathbf{k} \times \mathbf{p}_+)\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)} && \text{(integrated exponential)} && (38)\\
&= \frac{1}{\omega\mu_0}\mathbf{k} \times \mathbf{E}. && && (39)
\end{aligned}$$

(Remark for teaching assistants: This relation for plane waves should be known to the students by now from earlier exercises or the lecture, they may use it without derivation. Here we have shown that it is also valid for circular polarization.)

Now we compute the magnetic fields for the incoming, reflected and transmitted parts for one polarization:

$$\mathbf{H}_{\mathrm{i},+} = \frac{1}{\omega\mu_0}\mathbf{k}_\mathrm{i} \times \mathbf{E}_{\mathrm{i},+} \qquad \text{(by (39))} \tag{40}$$

$$= \frac{1}{\omega\mu_0}\mathbf{k}_\mathrm{i} \times \mathbf{p}_+ E_{\mathrm{i},+} \mathrm{e}^{\mathrm{i}(\mathbf{k}_i \mathbf{r} - \omega t)}. \qquad \text{(by (19))} \tag{41}$$

The reflected and transmitted parts can be computed analogously. Putting these magnetic fields into the continuity condition yields for the interface $\mathbf{r}_\parallel = (x, y, 0)$ the equation

$$\frac{1}{\omega\mu_0}(\mathbf{k}_\mathrm{i} \times \mathbf{p}_+ E_{\mathrm{i},+}\mathrm{e}^{\mathrm{i}\mathbf{k}_i \mathbf{r}_\parallel} + \mathbf{k}_\mathrm{r} \times \mathbf{p}_+ E_{\mathrm{r},+}\mathrm{e}^{\mathrm{i}\mathbf{k}_r \mathbf{r}_\parallel} - \mathbf{k}_{\mathrm{t},+} \times \mathbf{p}_+ E_{\mathrm{t},+}\mathrm{e}^{\mathrm{i}\mathbf{k}_{t,+} \mathbf{r}_\parallel})\mathrm{e}^{-\mathrm{i}\omega t} = 0. \tag{42}$$

All wave vectors involved are parallel to $\hat{\mathbf{e}}_z$ (due to the normal incidence), hence the spatial exponential parts are all equal to 1. Dividing by the frequency part and multiplying by $\omega\mu_0$:

$$\mathbf{k}_\mathrm{i} \times \mathbf{p}_+ E_{\mathrm{i},+} + \mathbf{k}_\mathrm{r} \times \mathbf{p}_+ E_{\mathrm{r},+} - \mathbf{k}_{\mathrm{t},+} \times \mathbf{p}_+ E_{\mathrm{t},+} = 0. \tag{43}$$

Enter the phase-jump at the interface: We have $\mathbf{k}_\mathrm{r} = -\mathbf{k}_\mathrm{i}$ and $\mathbf{k}_\mathrm{i} = k_i \hat{\mathbf{e}}_z$, as well as $\mathbf{k}_{\mathrm{t},+} = k_{\mathrm{t},+}\hat{\mathbf{e}}_z$. Plugging this into above equation gives

$$(\hat{\mathbf{e}}_z \times \mathbf{p}_+)(k_\mathrm{i} E_{\mathrm{i},+} - k_\mathrm{i} E_{\mathrm{r},+} - k_{\mathrm{t},+} E_{\mathrm{t},+}) = 0. \tag{44}$$

This equation can only be fulfilled if at least on of the factors in the product is 0, but the vector part is

$$\hat{\mathbf{e}}_z \times \mathbf{p}_+ = \hat{\mathbf{e}}_z \times (\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y) \tag{45}$$

$$= \underbrace{\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x}_{=\hat{\mathbf{e}}_y} + \mathrm{i}\underbrace{\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_y}_{-\hat{\mathbf{e}}_x} \tag{46}$$

$$= \hat{\mathbf{e}}_y + \mathrm{i}\hat{\mathbf{e}}_x \tag{47}$$

$$\neq 0. \tag{48}$$

Hence, the scalar part in the parentheses of (44) must be zero, giving

$$k_\mathrm{i} E_{\mathrm{i},+} - k_\mathrm{i} E_{\mathrm{r},+} - k_{\mathrm{t},+} E_{\mathrm{t},+} = 0. \tag{49}$$

Now we plug in the dispersion relations (16) for both sides of the interface (with $n = 1$ for air), giving

$$\frac{\omega}{c}E_{\mathrm{i},+} - \frac{\omega}{c}E_{\mathrm{r},+} - \frac{\omega}{c}n_+ E_{\mathrm{t},+} = 0. \tag{50}$$

Now we divide by $\frac{\omega}{c}$ and end up with the desired second equation for the electric field amplitudes, where the derivation for the $\mathbf{p}_-$ polarization is identical:

$$E_{\mathrm{i},\pm} - E_{\mathrm{r},\pm} = n_\pm E_{\mathrm{t},\pm}. \tag{51}$$

The solution of (32) and for given incoming amplitude $E_{\mathrm{i},\pm}$ is

$$E_{\mathrm{r},\pm} = \frac{1-n_\pm}{1+n_\pm} E_{\mathrm{i},\pm}, \tag{52}$$

$$E_{\mathrm{t},\pm} = \frac{1+n_\pm}{2} E_{\mathrm{i},\pm}. \tag{53}$$

Only the reflected part is needed now.

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**c)** The vectors $\mathbf{p}_\pm$ are orthogonal and the amplitude $\mathbf{E}_\mathrm{r}$ has been expressed in terms of these polarization vectors by the components $E_{\mathrm{r},\pm}$ in (31), by which $\mathbf{E}_\mathrm{r}$ is determined completely. Thus the reflected wave part is given as the sum of

$$\mathbf{E}_\mathrm{r} = (E_{\mathrm{r},+}\mathbf{p}_+ + E_{\mathrm{r},-}\mathbf{p}_-)\mathrm{e}^{\mathrm{i}(-k_\mathrm{i}z-\omega t)} \qquad \text{(by (31))} \tag{54}$$

$$= (\frac{1-n_+}{1+n_+}\mathbf{p}_+ E_{\mathrm{i},+} + \frac{1-n_-}{1+n_-}\mathbf{p}_- E_{\mathrm{i},-})\mathrm{e}^{\mathrm{i}(-k_\mathrm{i}z-\omega t)} \qquad \text{(by (52))} \tag{55}$$

$$= \frac{1}{2}E_\mathrm{i}\begin{pmatrix} \frac{1-n_+}{1+n_+} + \frac{1-n_-}{1+n_-} \\ \mathrm{i}(\frac{1-n_+}{1+n_+} - \frac{1-n_-}{1+n_-}) \end{pmatrix}\mathrm{e}^{\mathrm{i}(-k_\mathrm{i}z-\omega t)}. \qquad \text{(by (18) and (19))} \tag{56}$$

For the intensity its sensible to introduce the abbreviations

$$a := \frac{1-n_+}{1+n_+}, \quad b := \frac{1-n_-}{1+n_-}, \tag{57}$$

such that the reflected part is written as

$$|\mathbf{E}_\mathrm{r}|^2 = \frac{|E_\mathrm{i}|^2}{4}[(a+b)^2 + (a-b)^2] \qquad \text{(by (56) and (57))} \tag{58}$$

$$= \frac{|E_\mathrm{i}|^2}{4}(a^2 + 2ab + b^2 + a^2 - 2ab + b^2) \qquad \text{(expanded squared terms)} \tag{59}$$

$$= \frac{|E_\mathrm{i}|^2}{4} \cdot 2 \cdot (a^2 + b^2) \qquad \text{(collected like terms)} \tag{60}$$

$$= \frac{|E_\mathrm{i}|^2}{2}\left[\left(\frac{1-n_+}{1+n_+}\right)^2 + \left(\frac{1-n_-}{1+n_-}\right)^2\right] \qquad \text{(substituted (57))} \tag{61}$$

Dividing by $|\mathbf{E}_\mathrm{i}|^2$ yields the desired result (17), where by (14), the incoming plane wave fulfills $|\mathbf{E}_\mathrm{i}|^2 = |E_\mathrm{i}|^2$.

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**d)** The reflected wave is given by (56). Its polarization vector is complex valued, where the real part has a different magnitude as the imaginary part. Thus the reflected wave is elliptically polarized.

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— Hand in solutions in lecture on 04.06.2012 —