KARLSRUHER INSTITUT FÜR TECHNOLOGIE (KIT) SS2012
INSTITUT FÜR THEORETISCHE FESTKÖRPERPHYSIK
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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

Name:______________________________

Problem set 3 for the course "Theoretical Optics"
Sample Solutions

5 Polarization States Of Plane Waves

We want to investigate superposition states of different kinds of polarizations.

a) We consider two circularly polarized plane waves with opposite sense of rotation at $\mathbf{r} = 0$ ($E_0 \in \mathbb{R}$):

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}=0,t) &= E_0\big(\hat{\mathbf{e}}_x \cos(\omega t) + \hat{\mathbf{e}}_y \sin(\omega t)\big) \\ \mathbf{E}_2(\mathbf{r}=0,t) &= E_0\big(\hat{\mathbf{e}}_x \cos(\omega t+\phi) - \hat{\mathbf{e}}_y \sin(\omega t+\phi)\big) \end{aligned} \tag{1}$$

Show that at $\mathbf{r} = 0$ the superposition $\mathbf{E}(t) = \mathbf{E}_1(t) + \mathbf{E}_2(t)$ is a linearly polarized plane wave. [**3** Point(s)]

b) Show that an elliptically polarized plane wave at $\mathbf{r} = 0$,

$$\mathbf{E}(\mathbf{r}=0,t) = \hat{\mathbf{e}}_x A\cos(\omega t) + \hat{\mathbf{e}}_y B\sin(\omega t) \quad \text{with } A, B \in \mathbb{R}, \tag{2}$$

can be written as the superposition of a circularly polarized plane wave and a linearly polarized plane wave (also at $\mathbf{r} = 0$). [**2** Point(s)]

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a) In this problem, we use these trigonometric identities (see Bronstein, e.g.):

$$\sin(\alpha) + \sin(\beta) = 2\sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big), \tag{3}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big). \tag{4}$$

Substituting (1) into $\mathbf{E}(t)$ and collecting like vector entries yields

$$\begin{aligned}\mathbf{E}_1(t) + \mathbf{E}_2(t) &= E_0 \hat{\mathbf{e}}_x \underbrace{\Big[\cos(\omega t) + \cos(\omega t + \phi)\Big]}_{2\cos(\omega t + \phi/2)\ \cos(-\phi/2)} \\ &\quad + E_0 \hat{\mathbf{e}}_y \underbrace{\Big[\sin(\omega t) - \sin(\omega t + \phi)\Big]}_{2\cos(\omega t + \phi/2)\ \sin(-\phi/2)} \qquad (5) \\ &= 2E_0 \underbrace{\Big[\hat{\mathbf{e}}_x \cos(\phi/2) - \hat{\mathbf{e}}_y \sin(\phi/2)\Big]}_{\text{time-independent}} \ \underbrace{\cos(\omega t + \phi/2)}_{\text{polarization-independent}} \qquad (6)\end{aligned}$$

Thus, the polarization vector is determined by the angle $\phi/2$ in the vector parts. The polarization direction is constant in time (this is the linear polarization state). The E-field oscillates along this direction with frequency ω and phase shift $\phi/2$. This is depicted in Fig. 1 and Fig. 2.

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b) This result is obtained by adding a **zero-term**:

$$\begin{aligned}\mathbf{E}(\mathbf{r} = \mathbf{0}, t) &= \hat{\mathbf{e}}_x A \cos(\omega t) + \hat{\mathbf{e}}_y B \sin(\omega t) \qquad (7) \\ &= \hat{\mathbf{e}}_x A \cos(\omega t) \underbrace{+ \hat{\mathbf{e}}_y A \sin(\omega t) - \hat{\mathbf{e}}_y A \sin(\omega t)}_{=0} + \hat{\mathbf{e}}_y B \sin(\omega t) \qquad (8) \\ &= \underbrace{A\Big[\hat{\mathbf{e}}_x \cos(\omega t) + \hat{\mathbf{e}}_y \sin(\omega t)\Big]}_{\text{circularly polarized}} + \underbrace{(B - A)\hat{\mathbf{e}}_y \sin(\omega t)}_{\text{linearly polarized}}. \qquad (9)\end{aligned}$$

This is the superposition of two waves, one is circularly polarized with amplitude A and one is linearly polarized with amplitude $B - A$. Note that the polarization vector changes its orientation in space with time now (this is the circular polarization state, depicted in Fig. 1).

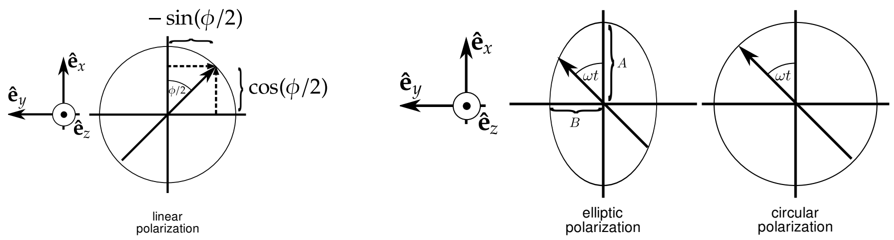

Figure 1: Polarization states as viewed *from* the positive z-axis. Left: The time independent polarization direction of part a). Note the minus sign, denoting the direction opposite to $\hat{\mathbf{e}}_y$. Right: The rotating polarization states, discernible by the time dependent angle ωt.

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6 Optical Tweezer[6 extra points]

The optical tweezer [L. Novotny et al., Phys.Rev.Lett., **79**, 645 (1997)] is one of the best examples for the application of the Maxwell Stress Tensor. We consider the force, which acts on a particle placed within a laser beam. For simplicity approximate the laser by a so-called

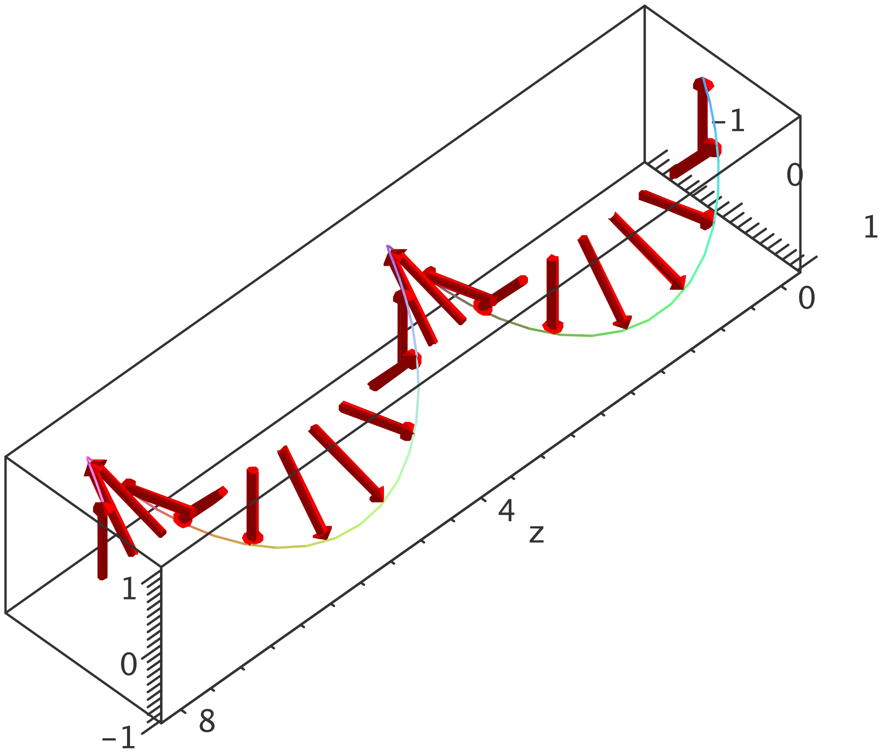

Figure 2: Electric field vectors along the z-axis for a circularly polarized plane wave at $t = 0$ propagating in positive z-direction. Animations of such waves can be found in the Maple worksheet that comes along with this solution or the Internet, e.g., Wikipedia. These may be shown in the problem session on a beamer.

Gaussian beam, for many experimental situations this is a reasonable well approximation. The electric field of a Gaussian beam is given in terms the radius r (radial distance from the center axis of the beam) and z (axial distance from the beam's narrowest point) in cylinder coordinates by the following expression:

$$E(r, z) = E_0 \frac{w_0}{w(z)} \exp\left(-\left(r/w(z)\right)^2\right) \exp\left(-ik\frac{r^2}{2R(z)}\right) \exp\left(i\left(\xi(z) - kz\right)\right)$$

where

$$\begin{aligned}
w_0 &- \text{minimum beam radius} \\
w(z) &= w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} - \text{beam radius} \\
z_0 &= \frac{\pi w_0^2}{\lambda} \\
R(z) &= z\left(1+\left(\frac{z_0}{z}\right)^2\right) \\
\xi(z) &= \arctan\left(\frac{z}{z_0}\right) \\
k &= \frac{2\pi}{\lambda}
\end{aligned}$$

Given the electric field, a simple way to approximate the force on a small particle (electric field assumed homogeneous over the size of the particle) is given by

$$\vec{F} = \frac{\alpha}{2}\vec{\nabla}E^2$$

where α is the particle's polarizability.

(i) Numerically calculate the force component in radial direction and visualize your result in a plot. Argue how your result explains the action of an optical tweezer.

A rigorous treatment, beyond the Rayleigh limitations above, yields the following expression for the force $\vec{F}$

$$\vec{F} = \int_{\partial V} \langle T \cdot \vec{n} \rangle \mathrm{d}S$$

where $\langle \ldots \rangle$ denotes a time average and T is the (electric) Maxwell stress tensor given by

$$T = \epsilon_0 \epsilon \vec{E}\vec{E} - \frac{1}{2}\epsilon_0 \epsilon E^2$$

(ii) For the above expression of the force, repeat your considerations from (i).

See Mathematica notebook file Tweezer.nb

— Hand in solutions in lecture on 21.05.2012 —