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Institut für theoretische Festkörperphysik
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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

Name:____

Problem set 2 for the course "Theoretical Optics" Sample Solutions

3 The energy carried by an EM wave travels with the energy velocity $v_\mathrm{e} := \langle S\rangle/\langle w\rangle$, where S denotes the magnitude of the Poynting vector and w the energy density of the wave. For waves with a definite frequency ω, the cycle-averaged mean values of these quantities in complex notation are given by

$$\langle w\rangle := \frac{1}{4}\mathrm{Re}\left(\mathbf{E}\mathbf{E}^*\frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega} + \mathbf{H}\mathbf{H}^*\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\right), \tag{1}$$

$$\langle S\rangle := \frac{1}{2}\left|\mathrm{Re}\big(\mathbf{E}\times\mathbf{H}^*\big)\right|. \tag{2}$$

Here, $\epsilon \equiv \epsilon_0\epsilon_r(\omega)$, $\mu \equiv \mu_0\mu_r(\omega)$, with real and positive $\epsilon_r(\omega)$ and $\mu_r(\omega)$.

Now, we consider a linearly polarized *modulated* plane wave

$$\mathbf{E}(\mathbf{r},t) = E_0(\mathbf{r},t)\hat{\mathbf{e}}_z\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)} \tag{3}$$

$$\text{with } \omega = \frac{k}{\sqrt{\epsilon\mu}}, \tag{4}$$

with a slowly varying envelope $E_0(\mathbf{r},t) \in \mathbb{C}$, such that we have $\mathbf{D}(\mathbf{r},t) = \epsilon(\omega)\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mu(\omega)\mathbf{H}(\mathbf{r},t)$.

a) Slowly varying means that E_0 does not change significantly on the length and time scales on which the term $\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$ oscillates. Explain in detail why, for this case, the derivatives of E_0 can be neglected relatively to the derivatives of $\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$ and find $\partial_t\mathbf{E}(\mathbf{r},t)$ and $\partial_{r_i}\mathbf{E}(\mathbf{r},t)$ in this approximation. *Remark:* This is known as the widely used slowly varying envelope approximation (SVEA). [**2** Point(s)]

b) Use Maxwell's equations to show that the corresponding magnetic field $\mathbf{H}(\mathbf{r},t)$ in the SVEA is given by an expression similar to (3) with $H_0(\mathbf{r},t)$ as the slowly varying amplitude of the magnetic field and determine the polarization direction of $\mathbf{H}(\mathbf{r},t)$. Show that $H_0(\mathbf{r},t) \propto \sqrt{\frac{\epsilon(\omega)}{\mu(\omega)}}E_0(\mathbf{r},t)$ and find the proper proportionality factor in the SVEA. With this magnetic field, express $\langle S\rangle$ and $\langle w\rangle$ in terms of the electric field envelope E_0. [**4** Point(s)]

c) Show that the energy velocity v_e can be expressed in terms of $\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\sqrt{\epsilon\mu})$. *Hint:* It is more straightforward to look at $\langle w\rangle/\langle S\rangle$ first. [**3** Point(s)]

d) Show that the energy velocity equals the group velocity $v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ of the wave. *Hint:* Recall the derivation rule for inverse functions to compute the inverse of the group velocity $1/v_\mathrm{g}$. [**1** Point(s)]

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a) We are given the plane wave

$$\mathbf{E}(\mathbf{r},t) = \underbrace{E_0(\mathbf{r},t)}_{f_\mathrm{e}}\,\hat{\mathbf{e}}_z\,\underbrace{\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}}_{f_\mathrm{c}} \tag{5}$$

with the envelope wave f_e and carrier wave f_c (for a simple example see Fig. 1). The full derivative is given by

$$\frac{\mathrm{d}(f_\mathrm{e} f_\mathrm{c})}{\mathrm{d}t} = \frac{\mathrm{d}f_\mathrm{e}}{\mathrm{d}t} f_\mathrm{c} + f_\mathrm{e}\frac{\mathrm{d}f_\mathrm{c}}{\mathrm{d}t}, \qquad \text{product rule} \tag{6}$$

where the first term shall be neglected. This is possible if

$$\left|\frac{\frac{\mathrm{d}f_\mathrm{e}}{\mathrm{d}t} f_\mathrm{c}}{f_\mathrm{e}\frac{\mathrm{d}f_\mathrm{c}}{\mathrm{d}t}}\right| = \frac{\left|\frac{\mathrm{d}f_\mathrm{e}}{\mathrm{d}t}\right|}{|f_\mathrm{e}||\omega|} \qquad (f_\mathrm{c} \text{ is plane wave, see (5)}) \tag{7}$$

$$\ll 1, \tag{8}$$

i.e., if the frequency ω of the carrier wave is so large that for all times we have

$$\boxed{\left|\frac{\mathrm{d}f_\mathrm{e}(t)}{\mathrm{d}t}\right| \ll |f_\mathrm{e}(t)||\omega|.} \text{ [1 Point(s)]} \tag{9}$$

The slowly varying envelope approximation (SVEA) then consists in omitting the small term $\frac{\mathrm{d}f_\mathrm{e}(t)}{\mathrm{d}t}$. In the example of Fig. 1, the SVEA works well for times, where the prerequisite (9) is fulfilled. For the spatial derivatives ∂_{r_i} , the values of the wavenumbers k_i (i.e., the oscillation frequency in space) for the corresponding directions have to be used instead of ω (i.e., the oscillation frequency in time).
Now, the total derivatives of the **E**-field are sums of one term merely depending on the carrier frequency ω or carrier wave number k and another term merely depending on the envelope change in time or space and hence being negligible as compared to the former ones. In particular, there are no mixed terms combining the carrier and the envelope derivatives. Applying the SVEA gives

$$\frac{\partial \mathbf{E}}{\partial t} = \left[\frac{\partial E_0}{\partial t} + E_0\cdot(-\mathrm{i}\omega)\right]\hat{\mathbf{e}}_z \exp\big(\mathrm{i}(\mathbf{kr}-\omega t)\big) \qquad \text{product rule} \tag{10}$$

$$\approx \boxed{-\mathrm{i}\omega E_0(\mathbf{r},t)\hat{\mathbf{e}}_z \exp\big(\mathrm{i}(\mathbf{kr}-\omega t)\big).} \text{ [0.5 Point(s)]} \qquad \text{SVEA by (9)} \tag{11}$$

The same argument can be applied to the spatial derivatives, with $\omega \mapsto k_i$ in (9), giving

$$\frac{\partial \mathbf{E}}{\partial r_i} = \left[\frac{\partial E_0}{\partial r_i} + E_0\cdot(\mathrm{i}k_i)\right]\hat{\mathbf{e}}_z \exp\big(\mathrm{i}(\mathbf{kr}-\omega t)\big) \qquad \text{product rule} \tag{12}$$

$$\approx \boxed{\mathrm{i}k_i E_0(\mathbf{r},t)\hat{\mathbf{e}}_z \exp\big(\mathrm{i}(\mathbf{kr}-\omega t)\big).} \text{ [0.5 Point(s)]} \qquad \text{SVEA by (9)} \tag{13}$$

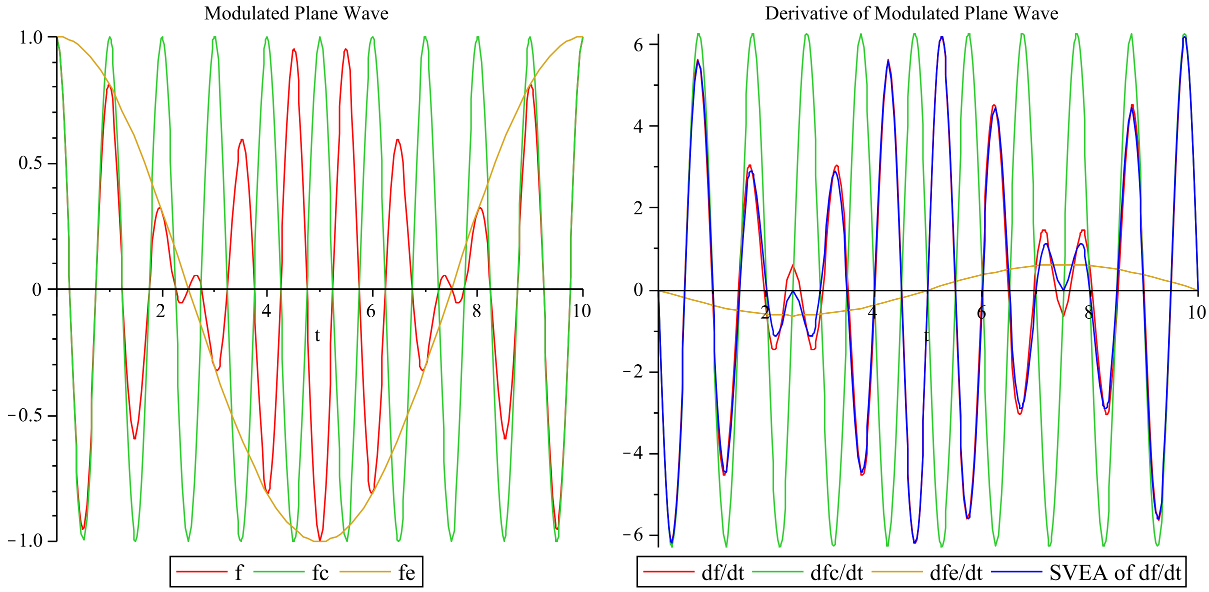

Figure 1: Example for the slowly varying envelope approximation (SVEA) for a harmonic modulation/envelope with period T on a harmonic carrier wave with period $10T$. Left: the full wave $f = f_\mathrm{e} f_\mathrm{c}$, the carrier wave f_c and the envelope function f_e. Right: First order derivatives with respect to time of the full wave f, the carrier wave f_c only, the envelope function f_e only and the SVEA $(\frac{\mathrm{d}f}{\mathrm{d}t})_\mathrm{SVEA} = f_\mathrm{e}\frac{\mathrm{d}f_\mathrm{c}}{\mathrm{d}t}$. Note the striking similarity of the approximation with the true derivative of f, except for those times, where the maximal absolute values $|\frac{\mathrm{d}f_\mathrm{e}}{\mathrm{d}t}|_\mathrm{max}$ coincide with the zero-crossings of the carrier wave derivative $\frac{\mathrm{d}f_\mathrm{c}}{\mathrm{d}t} = 0$. At these positions, the envelope apparently changes faster than the carrier, violating the prerequisite (9) for the SVEA. **However, the overall accuracy of this approximation justifies its usage.** For the spatial derivatives, the argument is the same. In practice, the SVEA is often used with broad gaussian pulses, where the prerequisite is never violated.

b) First we derive the desired relation for the magnetic field. By Maxwell's Equations, we have

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\boldsymbol{\nabla} \times \mathbf{E} && \text{Maxwell's equations} \quad (14)\\ &= -E_0 \boldsymbol{\nabla} \times \left(\hat{\mathbf{e}}_z \exp\left(\mathrm{i}(\mathbf{kr} - \omega t)\right)\right) && \text{by (13)} \quad (15)\\ &= \begin{pmatrix} -\mathrm{i}k_y \\ \mathrm{i}k_x \\ 0 \end{pmatrix} E_0 \exp\left(\mathrm{i}(\mathbf{kr} - \omega t)\right) && (16) \end{aligned}$$

We can choose the orientation of our coordinate system freely, thus we align our x-axis along $\mathbf{k}$, such that

$$\mathbf{k} = k\hat{\mathbf{e}}_x. \quad (17)$$

Now we insert the constitutive relation $\mathbf{B}(\mathbf{r}, t) = \mu(\omega)\mathbf{H}(\mathbf{r}, t)$ into (16). Note that μ

itself does *not* depend on time, such that we have

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{\mu}\frac{\partial \mathbf{B}}{\partial t} \tag{18}$$

$$= \frac{1}{\mu} \mathrm{i}k E_0 \hat{\mathbf{e}}_y \exp\big(\mathrm{i}(\mathbf{kr} - \omega t)\big) \qquad \text{by (16) and (17)} \tag{19}$$

$$\tag{20}$$

A simple integration combined with the SVEA (recall, E_0 still depends on time t) yields

$$\mathbf{H} = \frac{1}{\mu}\left(\frac{\mathrm{i}k}{-\mathrm{i}\omega}\right) E_0 \hat{\mathbf{e}}_y \exp\big(\mathrm{i}(\mathbf{kr} - \omega t)\big) \tag{21}$$

$$= -\frac{k}{\mu\omega} E_0 \hat{\mathbf{e}}_y \exp\big(\mathrm{i}(\mathbf{kr} - \omega t)\big) \tag{22}$$

$$= -\frac{\sqrt{\epsilon\mu}}{\mu} E_0 \hat{\mathbf{e}}_y \exp\big(\mathrm{i}(\mathbf{kr} - \omega t)\big) \qquad \text{by (4)} \tag{23}$$

$$= \sqrt{\frac{\epsilon_0\epsilon_\mathrm{r}}{\mu_0\mu_\mathrm{r}}} E_0 (-\hat{\mathbf{e}}_y) \exp\big(\mathrm{i}(\mathbf{kr} - \omega t)\big). \qquad \text{reduced fraction} \tag{24}$$

The minus sign is optically annoying and enters due to our choice of coordinates. A plane wave propagating along z and being polarized in x contains no minus sign. So, we define

$$\mathbf{H} := H_0(-\hat{\mathbf{e}}_y) \exp\big(\mathrm{i}(\mathbf{kr} - \omega t)\big), \tag{25}$$

and by (24) we then have

$$\boxed{H_0(\mathbf{r}, t) = \sqrt{\frac{\epsilon_0\epsilon_\mathrm{r}}{\mu_0\mu_\mathrm{r}}} E_0(\mathbf{r}, t), \text{ [2 Point(s)]}} \tag{26}$$

which is the desired relation between magnetic and electric field.

For the magnitude of the cycle-averaged Poynting vector, we plug (3) and (26) into (2), where the carrier wave part cancels out and we have

$$\langle S \rangle = \frac{1}{2}\Big|\mathrm{Re}\Big(\sqrt{\frac{\epsilon_0\epsilon_\mathrm{r}}{\mu_0\mu_\mathrm{r}}} \underbrace{E_0 E_0^*}_{|E_0|^2}\Big)\Big| \cdot \underbrace{|\hat{\mathbf{e}}_z \times (-\hat{\mathbf{e}}_y)|}_{=1} \tag{27}$$

$$= \boxed{\frac{1}{2}\sqrt{\frac{\epsilon_0\epsilon_\mathrm{r}}{\mu_0\mu_\mathrm{r}}} |E_0|^2, \text{ [1 Point(s)]}} \qquad (\epsilon_0, \mu_0, \epsilon_\mathrm{r}, \mu_\mathrm{r} \text{ are real and positive}) \tag{28}$$

For the cycle averaged energy density, we plug (3) and (26) into (1), where the carrier wave part cancels out again and we have

$$\langle w \rangle = \frac{1}{4}\mathrm{Re}\Big(|E_0|^2 \frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega} + |E_0|^2 \frac{\epsilon}{\mu}\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\Big) \tag{29}$$

$$= \boxed{\frac{1}{4}\Big(\frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega} + \frac{\epsilon}{\mu}\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\Big)|E_0|^2. \text{ [1 Point(s)]}} \qquad \text{collected E-field, all is real} \tag{30}$$

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c) We follow the hint and look at

$$\frac{1}{v_\mathrm{e}} = \frac{\langle w\rangle}{\langle S\rangle} \tag{31}$$

$$= \left[\frac{1}{4}\Big(\frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega} + \frac{\epsilon}{\mu}\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\Big)|E_0|^2\right] \Big/ \left[\frac{1}{2}\sqrt{\frac{\epsilon_0\epsilon_\mathrm{r}}{\mu_0\mu_\mathrm{r}}}|E_0|^2\right] \qquad \text{by (28) and (30)} \tag{32}$$

$$= \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\Big(\frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega} + \frac{\epsilon}{\mu}\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\Big) \qquad \text{reduced fraction} \tag{33}$$

$$= \frac{1}{2}\Big(\sqrt{\frac{\mu}{\epsilon}}(\epsilon + \omega\frac{\mathrm{d}\epsilon}{\mathrm{d}\omega}) + \sqrt{\frac{\epsilon}{\mu}}(\mu + \omega\frac{\mathrm{d}\mu}{\mathrm{d}\omega})\Big) \qquad \text{chain rule} \tag{34}$$

$$= \sqrt{\epsilon\mu} + \frac{\omega}{2}\Big(\sqrt{\frac{\mu}{\epsilon}}\frac{\mathrm{d}\epsilon}{\mathrm{d}\omega} + \sqrt{\frac{\epsilon}{\mu}}\frac{\mathrm{d}\mu}{\mathrm{d}\omega}\Big). \qquad \text{rearranged terms} \tag{35}$$

On closer inspection, this last equation turns out to be the derivative given in the problem text:

$$\frac{\mathrm{d}(\omega\sqrt{\epsilon\mu})}{\mathrm{d}\omega} = \sqrt{\epsilon\mu} + \frac{\omega}{2\sqrt{\epsilon\mu}}(\mu\frac{\mathrm{d}\epsilon}{\mathrm{d}\omega} + \epsilon\frac{\mathrm{d}\mu}{\mathrm{d}\omega}) \qquad \text{chain rule} \tag{36}$$

$$= \sqrt{\epsilon\mu} + \frac{\omega}{2}(\sqrt{\frac{\mu}{\epsilon}}\frac{\mathrm{d}\epsilon}{\mathrm{d}\omega} + \sqrt{\frac{\epsilon}{\mu}}\frac{\mathrm{d}\mu}{\mathrm{d}\omega}) \tag{37}$$

$$= \frac{1}{v_\mathrm{e}}. \qquad \text{by (59)} \tag{38}$$

Finally, this means that

$$\boxed{v_\mathrm{e} = \frac{1}{\frac{\mathrm{d}(\omega\sqrt{\epsilon\mu})}{\mathrm{d}\omega}},} \text{ [3 Point(s)]} \tag{39}$$

which is the wanted expression for the energy velocity.

d) Now, let's start with a function $f(x)$ and its inverse $f^{-1}(y)$, then we have

$$x = f^{-1}(f(x)). \tag{40}$$

Taking the derivative with respect to x on both sides yields

$$\underbrace{1}_{=\frac{\mathrm{d}x}{\mathrm{d}x}} = \frac{\mathrm{d}\Big(f^{-1}(f(x))\Big)}{\mathrm{d}x} \tag{41}$$

$$= \frac{\mathrm{d}\Big(f^{-1}(y)\Big)}{\mathrm{d}y}\Bigg|_{y=f(x)} \frac{\mathrm{d}\Big(f(x)\Big)}{\mathrm{d}x} \qquad \text{chain rule} \tag{42}$$

$$\Rightarrow \frac{1}{\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)} = \frac{\mathrm{d}(f^{-1})}{\mathrm{d}y}\Bigg|_{y=f(x)} \tag{43}$$

This means, the derivative of the inverse function is just the inverse of the derivative of the original function. Recall that $f^{-1}(y) = x$ and $y = f(x)$, so by changing the notation (not the math involved), we get

$$\frac{1}{\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)} = \frac{\mathrm{d}x}{\mathrm{d}f}, \tag{44}$$

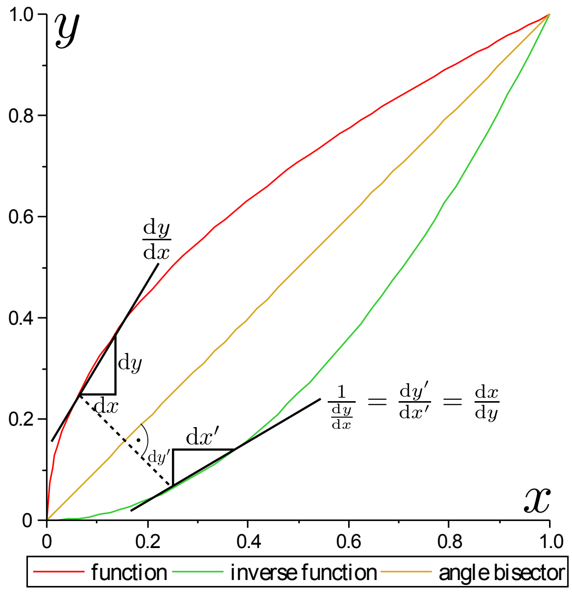

Figure 2: Example for derivatives of inverse functions. The inverse function is just given as the reflection at the angle bisector, and so is its slope. Here, $m = \frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of the function. The slope of the inverse function is $m' = \frac{\mathrm{d}y'}{\mathrm{d}x'}$, and by the reflection procedure we also see that $m' = 1/m$ and $\mathrm{d}x' = \mathrm{d}y$ and $\mathrm{d}y' = \mathrm{d}x$, hence $1/\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y}$, visualizing this derivation rule.

the desired relationship needed. The interpretation is easy (see Fig. 2). $\frac{\mathrm{d}f}{\mathrm{d}x}$ is the slope of the function f, meaning a small change $\mathrm{d}x$ in the variable amounts to a small change $\mathrm{d}f$ in the function values depending on x. However, if the inverse function exists, there is an unambiguous relation between f and x, such that interpreting f now as the independent variable, small change in $\mathrm{d}f$ defines a corresponding small change in $\mathrm{d}x$ that is needed to achieve that particular change in f. These two slopes are simply the inverses of each other, written out in a formula as (44).

Now we put it all together. For the group velovity $v_\mathrm{g} = \frac{\partial\omega}{\partial k}$ we need $\omega(k)$, which is implicitly given by (4), since the permittivity ϵ and the permeability μ also depend on ω. But we can write an *explicit* function $k(\omega)$:

$$k(\omega) = \omega\sqrt{\epsilon(\omega)\mu(\omega)} \qquad \text{by (4)} \tag{45}$$

Then we can calculate

$$\frac{1}{v_\mathrm{g}} = \frac{\mathrm{d}k}{\mathrm{d}\omega} \qquad \text{by (44)} \tag{46}$$

$$= \frac{\mathrm{d}\left(\omega\sqrt{\epsilon(\omega)\mu(\omega)}\right)}{\mathrm{d}\omega} \qquad \text{by (45)} \tag{47}$$

[1 Point(s)]

$$= \frac{1}{v_\mathrm{e}}, \qquad \text{by (59)} \tag{48}$$

hence we have

$$v_\mathrm{e} = v_\mathrm{g}, \tag{49}$$

which was to be shown. QED

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4 Effects Of Dispersion

We consider a Gaussian pulse with carrier frequency $\omega_0 = k_0 c$ in 1D given by

$$\mathbf{E}(x,t) = E_0 \hat{\mathbf{e}}_z \int \mathrm{d}k\, \mathrm{e}^{-\alpha(k-k_0)^2} \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}, \quad \alpha > 0, E_0 \in \mathbb{C}, \tag{50}$$

which propagates in a dispersive medium. For a wide pulse ($\alpha k_0^2 \gg 1$), the region of $\omega(k)$ around k_0 affects the wave propagation most significantly and we may approximate the dispersion relation by a truncated Taylor series

$$\omega(k) \simeq \omega_0 + \omega' \cdot (k - k_0) + \omega'' \cdot (k - k_0)^2, \tag{51}$$

where we used shorthand notations for the derivatives of the dispersion relation evaluated at k_0 as $\omega_0 = \omega(k_0)$, $\omega' = \omega'(k_0)$ and $\omega'' = \omega''(k_0)$.

a) Find the expression for the wave packet in terms of x and t by carrying out the integration over k in this approximation. [**3** Point(s)]
Hint: You will need the value of the integral $\int \mathrm{d}x\, \exp(ax^2 + bx), \mathrm{Re}(a) < 0$. With the help of the completion of squares, this can be recast into the integral $\int \mathrm{d}x\, \exp(a(x+z_0)^2)$ with value $\sqrt{\frac{\pi}{-a}}$, where z_0 is a complex number. Express (50) as an integral over $\kappa = k - k_0$ and apply this result.

b) In a Gaussian function $g(x) = A\mathrm{e}^{-(x-x_0)^2/(2\sigma^2)}$, we call A the peak amplitude, σ the pulse width and x_0 the peak position. We assume that we can use the slowly varying envelope approximation (SVEA) — You do not have to prove that. Then the pulse intensity $I(x,t) \propto |\mathbf{E}(x,t)|^2$. Interpret the behavior of the peak amplitude, pulse width and peak position of I in dispersive media for the electric field found in a). [**2** Point(s)]

c) For the solution found in a), us a computer algebra program of Your choice and plot the real part of the electric field (the physical wave) along with the modulus of the envelope at various times, such that the pulse broadening and pulse motion is clearly visible. Choose sensible values for the needed parameters and keep the broad pulse condition $\alpha k_0^2 \gg 1$ in mind. Based on these plots, explain that the SVEA is applicable here as discussed in problem 3a) — no calculations are necessary. Create the same plots for a narrow pulse and explain, why the SVEA fails then. What do You observe for the time evolution of the carrier wave here? [Remark: The solution of part a) was derived using the prerequisite $\alpha k_0^2 \gg 1$, so in principle we are not allowed to drop this condition all of a sudden. A real narrow pulse would have a slightly different function, since higher order derivatives of ω were necessary then. However, the principal features for such a pulse can be seen clearly here as well.] [**2 BONUS** Point(s)]

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a) Firstly, we need the **completion of squares which is basically the binomial identities backwards.** In particular, we apply this binomial identity

$$(A + B)^2 = A^2 + 2AB + B^2 \tag{52}$$

to the expression

$$a\kappa^2 + b\kappa = a(\underbrace{\kappa^2}_{=:A^2} + 2\underbrace{\kappa}_{=:A}\underbrace{\frac{b}{2a}}_{=:B} + \underbrace{\frac{b^2}{4a^2}}_{=:B^2}) - \frac{b^2}{4a} \qquad \text{added 0, rearranged terms} \tag{53}$$

$$= a(\kappa + \frac{b}{2a})^2 - \frac{b^2}{4a}. \qquad \text{by (52)} \tag{54}$$

Secondly, to evaluate the integral (50), we enter the dispersion relation $\omega(k)$ by the approximation (51) and perform a **substitution of variables** in the same step by defining a new integration variable κ (which is just k shifted by a constant k_0):

$$\kappa := k - k_0 \Longrightarrow k = \kappa + k_0, \quad \mathrm{d}k = \mathrm{d}\kappa. \tag{55}$$

Applying this substitution to the dispersion relation (51) gives

$$\omega(k) \approx \omega_0 + \omega' \cdot \kappa + \omega'' \cdot \kappa^2. \tag{56}$$

This result and the substitution (55) put in the integral (50) yields

$$\mathbf{E}(x,t) \approx E_0\hat{\mathbf{e}}_z \int \mathrm{d}\kappa \, \exp\Big(- \alpha\kappa^2 + \mathrm{i}\kappa x + \mathrm{i}k_0 x - \mathrm{i}\omega_0 t - \mathrm{i}\omega' t\kappa - \mathrm{i}\omega'' t\kappa^2\Big) \tag{57}$$

collecting powers of κ gives

$$= E_0\hat{\mathbf{e}}_z \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} \int \mathrm{d}\kappa \, \exp\Big(\underbrace{-(\alpha + \mathrm{i}\omega'' t)}_{=a}\kappa^2 + \underbrace{\mathrm{i}(x - \omega' t)}_{=b}\kappa\Big) \tag{58}$$

applying the completion of squares (54) yields

$$= E_0\hat{\mathbf{e}}_z \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} \exp(-\frac{b^2}{4a}) \underbrace{\int \mathrm{d}\kappa \, \exp\Big(a(\kappa + \frac{b}{2a})^2\Big)}_{=\sqrt{\frac{\pi}{-a}} \text{ after hint}}. \tag{59}$$

Resubstituting a, b and κ into (59) finally yields the electric field as

$$\boxed{\mathbf{E}(x,t) = \sqrt{\frac{\pi}{\alpha + \mathrm{i}\omega'' t}} E_0\hat{\mathbf{e}}_z \exp\Big(\mathrm{i}(k_0 x - \omega_0 t)\Big) \exp\Big(- \frac{(x - \omega' t)^2}{4(\alpha + \mathrm{i}\omega'' t)}\Big).} \tag{60}$$

[**3** Point(s)]

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b) The intensity $I(x,t)$ according to the SVEA is

$$I \propto |\mathbf{E}|^2 \tag{61}$$

$$= \frac{|E_0|^2\pi}{|\alpha + \mathrm{i}\omega'' t|}\Big| \exp\Big(- \frac{(x - \omega' t)^2}{4(\alpha + \mathrm{i}\omega'' t)}\Big)\Big|^2 \qquad \text{by (60)} \tag{62}$$

all variables are real, and expand the fraction in the exponential:

$$= \frac{|E_0|^2\pi}{\sqrt{\alpha^2 + (\omega'' t)^2}}\Big| \exp\Big(- \frac{(x - \omega' t)^2(\alpha - \mathrm{i}\omega'' t)}{4(\alpha + \mathrm{i}\omega'' t)(\alpha - \mathrm{i}\omega'' t)}\Big)\Big|^2 \tag{63}$$

exand the exponential

$$= \frac{|E_0|^2\pi}{\sqrt{\alpha^2+(\omega''t)^2}}\Big|\exp\Big(-\frac{\alpha(x-\omega't)^2}{4(\alpha^2+(\omega''t)^2)}\Big)\Big|^2 \underbrace{\Big|\exp\Big(-\mathrm{i}\frac{\omega''t(x-\omega't)^2}{4(\alpha^2+(\omega''t)^2)}\Big)\Big|^2}_{=1} \tag{64}$$

real exponentials are positive, and expand the exponential fraction by $1/\alpha$

$$= \frac{|E_0|^2\pi}{\sqrt{\alpha^2+(\omega''t)^2}}\exp\Big(\frac{(x-\omega't)^2}{2(\alpha+(\omega''t)^2/\alpha)}\Big). \tag{65}$$

Comparing this with the general form of a Gaussian function, we find

$$x_0 = \omega' t, \tag{66}$$

$$\sigma = \sqrt{\alpha + \frac{(\omega''t)^2}{\alpha}}, \tag{67}$$

[**1** Point(s)]

$$A \propto \frac{1}{\sqrt{\alpha^2+(\omega''t)^2}} \qquad \Big(= \frac{1}{\sqrt{\alpha}\sigma}\Big). \tag{68}$$

This result has the following interpretations (see also Fig. 3):

* The center x_0 of the pulse intensity moves uniformly (with constant velocity) with the group velocity $\omega' = \frac{\partial\omega}{\partial k}|_{k=k_0}$.
* The pulse width σ of the pulse is initially solely given by α (at $t = 0$, makes sense) and broadens with time depending on ω''.
* The pulse amplitude decreases in time as the width increases (also depending on ω'').

[**1** Point(s)]

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c) The plots in Fig. 4 show the time evolution of the broad pulse. The SVEA is applicable, because the envelope function rises much slower in space (has a flat slope) than the underlying carrier wave that oscillates rapidly with $\omega_0 = ck_0$, for $c = 1$ this is $\omega_0 = k_0$. The same is true for the time derivative, if the uniform motion ω' and the speed of the pulse broadening ω'' are also much smaller than ω_0.

On the other hand, the narrow pulse is shown in Fig. 5. The derivative of envelope and carrier waves are of the same order, so the SVEA does not apply here. When plotting this short pulse for various times t, one can see that the carrier oscillations become more rapid at the pulse front than compared to the tail. This phenomenon is called chirp and the pulse is said to be a chirped pulse. This phenomenon is also due to dispersion.

— Hand in solutions in lecture on 14.05.2012 —

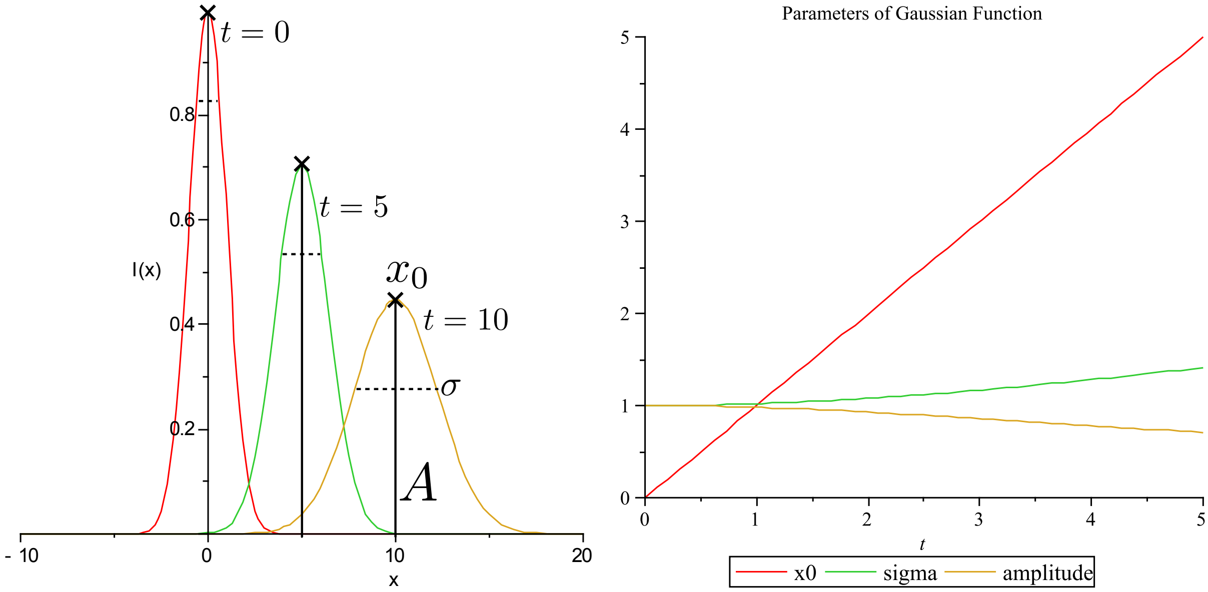

Figure 3: Example for the (intensity) pulse broadening in dispersive media after (65). For this example, the values $\omega' = 1$, $\omega'' = 0.2$ and $\alpha = 1$ were chosen. Left: The Gaussian pulse in space for different times t. The center x_0 moves uniformly, while the pulse broadens and decreases in amplitude. Right: The defining parameters $A(t)$, $x_0(t)$ and $\sigma(t)$ of the pulse and their time evolution.

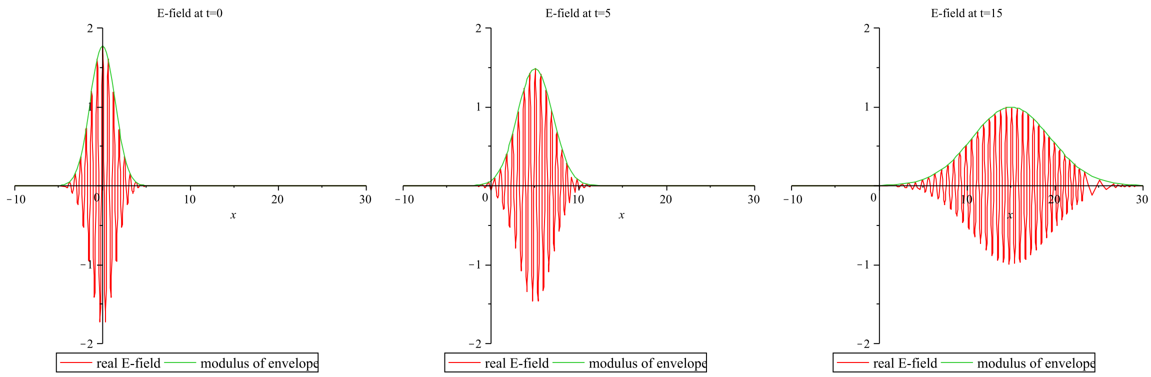

Figure 4: Real part of E_z-field (60) and modulus of envelope for the parameters $E_0 = \alpha = c = 1$, $k_0 = \omega_0 = 10$, $\omega' = 1$, $\omega'' = 0.2$.

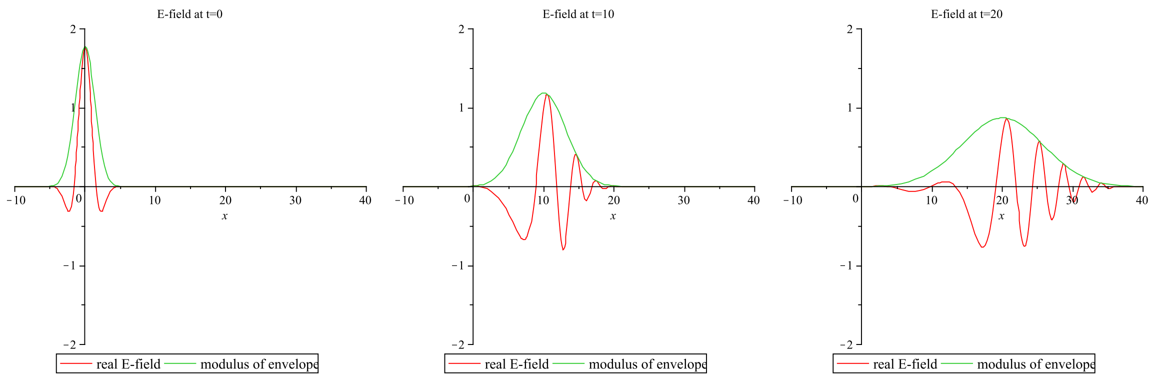

Figure 5: Same as Fig. 4, but with $\alpha = k_0 = 1$ (broad pulse condition violated). Note the chirp at advanced times t (see text).