Karlsruher Institut für Technologie (KIT) SS2012
Institut für theoretische Festkörperphysik
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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

☐ **Group 4.**

Name:______________________________

Problem set 2 for the course "Theoretical Optics"

3 Energy Velocity Of EM Waves In Matter

The energy carried by an EM wave travels with the energy velocity $v_\mathrm{e} := \langle S\rangle/\langle w\rangle$, where S denotes the magnitude of the Poynting vector and w the energy density of the wave. For waves with a definite frequency ω, the cycle-averaged mean values of these quantities in complex notation are given by

$$\langle w\rangle := \frac{1}{4}\mathrm{Re}\left(\mathbf{E}\mathbf{E}^*\frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega} + \mathbf{H}\mathbf{H}^*\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\right), \tag{1}$$

$$\langle S\rangle := \frac{1}{2}\left|\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right)\right|. \tag{2}$$

Here, $\epsilon \equiv \epsilon_0\epsilon_r(\omega)$, $\mu \equiv \mu_0\mu_r(\omega)$, with real and positive $\epsilon_r(\omega)$ and $\mu_r(\omega)$.

Now, we consider a linearly polarized *modulated* plane wave

$$\mathbf{E}(\mathbf{r},t) = E_0(\mathbf{r},t)\hat{\mathbf{e}}_z\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)} \tag{3}$$

$$\text{with } \omega = \frac{k}{\sqrt{\epsilon\mu}}, \tag{4}$$

with a slowly varying envelope $E_0(\mathbf{r},t) \in \mathbb{C}$, such that we have $\mathbf{D}(\mathbf{r},t) = \epsilon(\omega)\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mu(\omega)\mathbf{H}(\mathbf{r},t)$.

a) Slowly varying means that E_0 does not change significantly on the length and time scales on which the term $\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$ oscillates. Explain in detail why, for this case, the derivatives of E_0 can be neglected relatively to the derivatives of $\mathrm{e}^{\mathrm{i}(\mathbf{kr}-\omega t)}$ and find $\partial_t\mathbf{E}(\mathbf{r},t)$ and $\partial_{r_i}\mathbf{E}(\mathbf{r},t)$ in this approximation. *Remark:* This is known as the widely used slowly varying envelope approximation (SVEA). [**2** Point(s)]

b) Use Maxwell's equations to show that the corresponding magnetic field $\mathbf{H}(\mathbf{r},t)$ in the SVEA is given by an expression similar to (3) with $H_0(\mathbf{r},t)$ as the slowly varying amplitude of the magnetic field and determine the polarization direction of $\mathbf{H}(\mathbf{r},t)$. Show that $H_0(\mathbf{r},t) \propto \sqrt{\frac{\epsilon(\omega)}{\mu(\omega)}}E_0(\mathbf{r},t)$ and find the proper proportionality factor in the

SVEA. With this magnetic field, express $\langle S \rangle$ and $\langle w \rangle$ in terms of the electric field envelope E_0. [**4** Point(s)]

c) Show that the energy velocity v_e can be expressed in terms of $\frac{\mathrm{d}}{\mathrm{d}\omega}(\omega\sqrt{\epsilon\mu})$. *Hint:* It is more straightforward to look at $\langle w \rangle / \langle S \rangle$ first. [**3** Point(s)]

d) Show that the energy velocity equals the group velocity $v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ of the wave. *Hint:* Recall the derivation rule for inverse functions to compute the inverse of the group velocity $1/v_\mathrm{g}$. [**1** Point(s)]

4 Effects Of Dispersion

We consider a Gaussian pulse with carrier frequency $\omega_0 = k_0 c$ in 1D given by

$$\mathbf{E}(x,t) = E_0 \hat{\mathbf{e}}_z \int \mathrm{d}k\, \mathrm{e}^{-\alpha(k-k_0)^2} \mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}, \quad \alpha > 0, E_0 \in \mathbb{C}, \tag{5}$$

which propagates in a dispersive medium. For a wide pulse ($\alpha k_0^2 \gg 1$), the region of $\omega(k)$ around k_0 affects the wave propagation most significantly and we may approximate the dispersion relation by a truncated Taylor series

$$\omega(k) \simeq \omega_0 + \omega' \cdot (k-k_0) + \omega'' \cdot (k-k_0)^2, \tag{6}$$

where we used shorthand notations for the derivatives of the dispersion relation evaluated at k_0 as $\omega_0 = \omega(k_0)$, $\omega' = \omega'(k_0)$ and $\omega'' = \omega''(k_0)$.

a) Find the expression for the wave packet in terms of x and t by carrying out the integration over k in this approximation. [**3** Point(s)]
Hint: You will need the value of the integral $\int \mathrm{d}x\, \exp(ax^2+bx), \mathrm{Re}(a) < 0$. With the help of the completion of squares, this can be recast into the integral $\int \mathrm{d}x\, \exp(a(x+z_0)^2)$ with value $\sqrt{\frac{\pi}{-a}}$, where z_0 is a complex number. Express (5) as an integral over $\kappa = k - k_0$ and apply this result.

b) In a Gaussian function $g(x) = A\mathrm{e}^{-(x-x_0)^2/(2\sigma^2)}$, we call A the peak amplitude, σ the pulse width and x_0 the peak position. We assume that we can use the slowly varying envelope approximation (SVEA) — You do not have to prove that. Then the pulse intensity $I(x,t) \propto |\mathbf{E}(x,t)|^2$. Interpret the behavior of the peak amplitude, pulse width and peak position of I in dispersive media for the electric field found in a). [**2** Point(s)]

c) For the solution found in a), us a computer algebra program of Your choice and plot the real part of the electric field (the physical wave) along with the modulus of the envelope at various times, such that the pulse broadening and pulse motion is clearly visible. Choose sensible values for the needed parameters and keep the broad pulse condition $\alpha k_0^2 \gg 1$ in mind. Based on these plots, explain that the SVEA is applicable here as discussed in problem 3a) — no calculations are necessary. Create the same plots for a narrow pulse and explain, why the SVEA fails then. What do You observe for the time evolution of the carrier wave here? [Remark: The solution of part a) was derived using the prerequisite $\alpha k_0^2 \gg 1$, so in principle we are not allowed to drop this condition all of a sudden. A real narrow pulse would have a slightly different function, since higher order derivatives of ω were necessary then. However, the principal features for such a pulse can be seen clearly here as well.] [**2 BONUS** Point(s)]

— Hand in solutions in tutorial on 14.05.2012 —