Karlsruher Institut für Technologie (KIT) SS2012
Institut für theoretische Festkörperphysik

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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

Name:____

Problem set 1 for the course "Theoretical Optics"
Sample Solutions

1 Space Probe Propulsion By Solar Sails

We idealize a solar sail of a space probe by an ideal planar mirror (i.e. no losses due to absorption/heat generation) of rest mass m (Fig. 1). Let W_i be the total amount of energy of an incoming localized wave packet at normal incidence to the plane mirror. The wave packet shall be reflected by the mirror and have a total amount of energy W_r after the reflection.

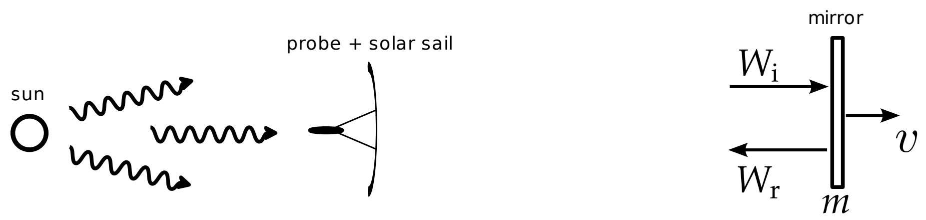

Figure 1: Realistic and idealized model for a solar sail propelled space probe.

a) Assume that the mirror is at rest before the reflection process. Using the relativistic formulae for energy and momentum, write down explicitly the expressions for the conservation of momentum and of energy for the reflection process, respectively. Then, show that the velocity v of the mirror and the energy W_r of the reflected wave packet can be written as

$$v = c\frac{(1+2B)^2-1}{(1+2B)^2+1}, \qquad W_\mathrm{r} = W_\mathrm{i}\frac{1}{1+2B}, \tag{1}$$

and determine the constant B in terms of m and W_i. [**4** Point(s)]

b) Find expressions for W_r and v in the limits $W_\mathrm{i} \ll mc^2$ and $W_\mathrm{i} \gg mc^2$: Identify the parameter ϵ which is much smaller than 1 in these cases and apply a Taylor series expansion up to the leading order in ϵ (i.e., beyond the $0th$ order). [**3** Point(s)]

Our sun emits an instantaneous power (luminosity) of $P_\mathrm{sun} = \frac{dW}{dt} \approx 3.8 \times 10^{26}\,\mathrm{J/s}$. Now consider an actual space probe with a sail of area A and assume that the weight of the probe is solely determined by its sail, which consists of a material with density $1\,\mathrm{g/m^2}$.

c) First, determine which of the limits (if any) discussed in (b) is appropriate if the probe is launched at the earth (distance r to the sun $= 1\,\mathrm{AU} \approx 1.5 \times 10^{11}\,\mathrm{m}$). Use the appropriate limit to estimate the pressure on the sail as a function of the distance from the sun. Show that this pressure is given by an expression of the form

$$\frac{C}{r^2} \tag{2}$$

and determine the constant C. [**2** Point(s)]

d) With the pressure from (c), first write down the mechanical equation of motion of the probe. Since this equation can not be solved in closed form, make the crude approximation that the pressure remains constant at the value it has for $x = 1\,\mathrm{AU}$. Further assume a sail with area $A = 100 \times 100\,\mathrm{m}$. After which time would the probe leave our solar system ($\approx 250\,\mathrm{AU}$) and when could it reach Alpha Centauri ($2.8 \times 10^5\,\mathrm{AU}$)? Which velocities does the probe have then? Is the non-relativistic treatment still valid when the probe reached the respective distances? [**2** Point(s)]

e) Use a numerical method (e.g., the ordinary differential equation solvers built into Maple, Mathematica, Matlab, etc.) to solve the full differential equation from (d). Assume that the probe is launched from earth and is in rest when the sail is expanded. Plot the distance of the probe and its velocity as a function of time and compare with the result from (d). [**2 BONUS** Point(s)]

We start from the following expressions (m is the rest mass):

$$E = m\gamma c^2 \tag{3}$$

$$p = m\gamma v \tag{4}$$

$$E^2 = (pc)^2 + \left(mc^2\right)^2 \tag{5}$$

Here, the γ-factor is given by

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{6}$$

where $\beta = \frac{v}{c}$ is the velocity in units of the speed of light c. Thus, for light where $m = 0$, the momentum is given by $p = E/c$.

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a) **Conservation of energy** gives the expression (total energy before process equals total energy after process)

$$\boxed{W_\mathrm{i} + mc^2 = W_\mathrm{r} + m\gamma c^2} \text{ [1 Point(s)]} \tag{7}$$

while **conservation of momentum** yields (total momentum before process equals total momentum after process)

$$\boxed{\frac{W_\mathrm{i}}{c} = -\frac{W_\mathrm{r}}{c} + m\gamma v.} \text{ [1 Point(s)]} \tag{8}$$

Scaling (8) with c and adding it to (7) yields

$$\begin{aligned} 2W_\mathrm{i} + mc^2 &= m\gamma c^2 + m\gamma c v \\ &= m\gamma c^2 (1+\beta) && \text{subs } \beta = v/c \\ &= mc^2 \frac{1+\beta}{\sqrt{1-\beta^2}} && \text{subs } \gamma \text{ by (6)} \\ &= mc^2 \frac{\sqrt{1+\beta}\sqrt{1+\beta}}{\sqrt{1+\beta}\sqrt{1-\beta}} && \text{expand fraction} \\ &= mc^2 \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} && \text{reduce fraction} \end{aligned}$$

Thus, we find

$$\underbrace{\left(1 + 2\frac{W_\mathrm{i}}{mc^2}\right)^2}_{=:a} = \frac{1+\beta}{1-\beta}, \tag{9}$$

which we can solve for β as

$$\boxed{\beta = \frac{a-1}{a+1} = \frac{\left(1+2\frac{W_\mathrm{i}}{mc^2}\right)^2 - 1}{\left(1+2\frac{W_\mathrm{i}}{mc^2}\right)^2 + 1}.} \text{ [1 Point(s)]} \tag{10}$$

(Remark for teaching assistants: Alternative forms of this solution include

$$\begin{aligned} v &= \frac{(2W_\mathrm{i} + mc^2)^2 - m^2c^4}{(2W_\mathrm{i} + mc^2)^2 + m^2c^4} \\ &= \frac{2W_\mathrm{i} mc^3 + 2W_\mathrm{i}^2 c}{m^2c^4 + 2W_\mathrm{i} mc^2 + 2W_\mathrm{i}^2} \\ &= \frac{2W_\mathrm{i} c(mc^2 + W_\mathrm{i})}{(mc^2 + W_\mathrm{i})^2 + W_\mathrm{i}^2} \\ &= \sqrt{1 - \frac{1}{[1 + \frac{W_\mathrm{i}}{mc^2(1+\frac{mc^2}{2W_\mathrm{i}})}]^2}} \\ &= \sqrt{1 - \frac{1}{[1 + \frac{W_\mathrm{i}}{mc^2} - \frac{W_\mathrm{i}}{2W_\mathrm{i}+mc^2}]^2}}.) \end{aligned}$$

The result is an expression for the velocity of the space probe as a function of the energy of the incoming light.

Similarly, by scaling (8) with c and subtracting it from (7), we obtain

$$mc^2 = 2W_\mathrm{r} + m\gamma c^2 (1-\beta), \tag{11}$$

which we can solve for W_r:

$$\begin{aligned}
W_\mathrm{r} &= \frac{1}{2}mc^2\left(1-\gamma(1-\beta)\right) \\
&= \frac{1}{2}mc^2\left(1-\frac{1-\beta}{\sqrt{1-\beta^2}}\right) && \text{subs } \gamma \text{ by (6)} \\
&= \frac{1}{2}mc^2\left(1-\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\right) && \text{reduce fraction} \\
&= \frac{1}{2}mc^2\left(1-\frac{1}{\sqrt{a}}\right) && \text{subs } a \text{ by (9)} \\
&= \frac{1}{2}mc^2\left(1-\frac{1}{1+2\frac{W_\mathrm{i}}{mc^2}}\right) && \text{subs } a \text{ by (9)} \\
&= \frac{1}{2}mc^2\left(\frac{2\frac{W_\mathrm{i}}{mc^2}}{1+2\frac{W_\mathrm{i}}{mc^2}}\right) && \text{normal form of fraction} \\
&= \boxed{W_\mathrm{i}\frac{1}{1+2\frac{W_\mathrm{i}}{mc^2}}.}\text{ [1 Point(s)]} && \text{reduce fraction}
\end{aligned}$$

(Remark for teaching assistants: Alternative forms of this solution include

$$\begin{aligned}
W_\mathrm{r} &= \frac{W_\mathrm{i}mc^2}{2W_\mathrm{i}+mc^2} \\
&= W_\mathrm{i}-\frac{(2W_\mathrm{i})^2}{2(2W_\mathrm{i}+mc^2)}.)
\end{aligned}$$

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b) $W_\mathrm{i} \ll mc^2$: This is equivalent to $\boxed{\epsilon := \frac{W_\mathrm{i}}{mc^2} \ll 1}$ [**0.5** Point(s)]. Thus, we do a **Taylor series expansion** in ϵ:

$$\boxed{\beta = \frac{(1+2\epsilon)^2-1}{(1+2\epsilon)^2+1} = 2\epsilon + O(\epsilon^2) \approx 0} \text{ [0.5 Point(s)]} \tag{12}$$

$$\boxed{\frac{W_\mathrm{r}}{W_\mathrm{i}} = \frac{1}{1+2\epsilon} = 1-2\epsilon+O(\epsilon^2) \approx 1} \text{ [0.5 Point(s)]} \tag{13}$$

If the rest energy of the sail is much larger than the energy of the incoming light, then there is almost no acceleration. Almost all energy is reflected and the velocity of the sail is very small.

$W_\mathrm{i} \gg mc^2$: This is equivalent to $\boxed{\epsilon := \frac{mc^2}{W_\mathrm{i}} \ll 1}$ [**0.5** Point(s)]. Thus, we do a series expansion in ϵ:

$$\boxed{\beta = \frac{\left(1+2\frac{1}{\epsilon}\right)^2-1}{\left(1+2\frac{1}{\epsilon}\right)^2+1} = 1-\frac{\epsilon^2}{2}+O(\epsilon^3) \approx 1} \text{ [0.5 Point(s)]} \tag{14}$$

$$\boxed{\frac{W_\mathrm{r}}{W_\mathrm{i}} = \frac{1}{1+2\frac{W_\mathrm{i}}{mc^2}} = \frac{\epsilon}{2}+O(\epsilon^2) \approx 0} \text{ [0.5 Point(s)]} \tag{15}$$

If the rest energy of the sail is much smaller compared to the energy of the incoming light, then almost the entire energy of the light is used for acceleration. Only very little energy is reflected and the velocity of the sail approaches the velocity of light.

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c) (Remark for teaching assistants: This exercise is hard to rate adequately. No point suggestions are made here, but at least the proper limit to use should be explained properly by the students.) The mass of our probe can be expressed by the density ρ and the sail area A as $m = \rho A$, so that the rest energy is given by $E_0 = \rho A c^2$. The energy from the sun which reaches the sail per time T is given by $E_{\text{sun}} = A\frac{P_{\text{sun}}T}{4\pi r^2}$ (actually, an approximation for large distances entered here, see Fig. 2). The ratio between the

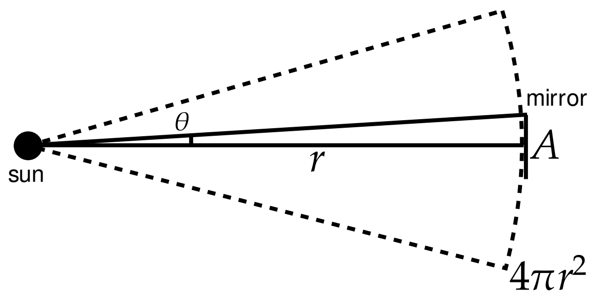

Figure 2: Mirror area A compared to sphere surface area $4\pi r^2$. The solid angle occupied by the mirror is given by the projection of the mirror area to the sphere surface area. For large distances r and small angles θ, this projection is in good approximation given by $A/4\pi r^2$.

energies is then given as

$$\frac{E_0}{E_{\text{sun}}} = \frac{\rho A c^2}{A\frac{P_{\text{sun}}T}{4\pi r^2}} = \frac{4\pi r^2 \rho c^2}{P_{\text{sun}}}\frac{1}{T} \approx 6.69 \times 10^{10}\text{s}\frac{1}{T}, \tag{16}$$

where all numerical values given in the problem have been entered. We find that, even when integrating the energy for a year ($\approx 3.15\times 10^7$ s), the rest energy of the sail is much larger than the energy of the light. This justifies to work in the $\boxed{\text{non-relativistic limit}}$, where by (13) we have

$$\frac{W_\text{r}}{W_\text{i}} \approx 1. \tag{17}$$

The pressure can be calculated in various ways. Here, we obtain the pressure u from the force per area, so $u = F/A$. To obtain the force, we use Newtons law that the force is given by the temporal change of momentum, so

$$u = F/A = \frac{1}{A}\frac{\partial p}{\partial t}. \tag{18}$$

As we have seen in problem 1a), the momentum of the probe is simply given by

$$p = \frac{W_\text{i} + W_\text{r}}{c} \approx \frac{2W_\text{i}}{c}. \qquad \text{by (17)} \tag{19}$$

The force is then given by

$$F = \frac{\partial p}{\partial t} \approx \frac{2}{c}\frac{\partial W_\mathrm{i}}{\partial t} = \frac{2}{c}A\frac{P_\mathrm{sun}}{4\pi r^2}. \tag{20}$$

So, the pressure is given by

$$u = F/A = \frac{P_\mathrm{sun}}{2\pi c}\frac{1}{r^2} = 8.96 \times 10^{-6}\,\mathrm{Pascal}. \tag{21}$$

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d) Since we are working in the non-relativisitic limit, we can write **Newtons law of motion** (as opposed to relativistic motion involving γ)

$$m\frac{\partial^2 r}{\partial t^2} = \rho A\frac{\partial^2 r}{\partial t^2} = F, \tag{22}$$

so by (21) we have

$$\boxed{\frac{\partial^2 r}{\partial t^2} = \frac{P_\mathrm{sun}}{2\pi c\rho}\frac{1}{r^2}.}\ [\mathbf{0.5}\ \text{Point(s)}] \tag{23}$$

As the problem text suggested, assuming a constant acceleration $a_0 = \frac{P_\mathrm{sun}}{2\pi c\rho}\frac{1}{x^2}$, we can solve this equation of motion as

$$r(t) = r_0 + v_0 t + \frac{a_0}{2}t^2, \tag{24}$$

where $v_0 = 0$ and $r_0 = x = 1\,\mathrm{AU}$. Solving this equation for a given distance r yields:

$$\boxed{t = \sqrt{2\frac{r - r_0}{a_0}},}\ [\mathbf{0.5}\ \text{Point(s)}] \tag{25}$$

so we obtain:

$$t_\mathrm{SolarSystem} \approx 9.13 \times 10^7\mathrm{s} = 2.89\mathrm{y} \tag{26}$$

$$t_\mathrm{AlphaCentauri} \approx 3.1 \times 10^9\mathrm{s} = 97\mathrm{y} \tag{27}$$

Makes sense to also check the velocity to make sure, our assumption of non-relativistic calculations are still okay:

$$\boxed{v(t) = a_0 t = \sqrt{2a_0(r - r_0)},}\ [\mathbf{0.5}\ \text{Point(s)}] \tag{28}$$

so

$$v_\mathrm{SolarSystem} \approx 8.18 \times 10^5\mathrm{m/s} \approx 0.0027c \tag{29}$$

$$v_\mathrm{AlphaCentauri} \approx 2.7 \times 10^7\mathrm{m/s} \approx 0.1c \tag{30}$$

(Remark for teaching assistants: Students will most likely say, $0.1c$ is not much smaller than c, only smaller, and hence the relativistic treatment is no longer allowed. You can give points for that. However, in the problem session, they should be taught the following interpretation.) The difference between Einstein (relativistic) and Newton (classic) lies in the form of the equations of motion, to behold for the velocities when the force F is given:

$$F = m\frac{\partial}{\partial t}v \quad \text{(Newton)}, \qquad F = m\frac{\partial}{\partial t}(\gamma v) = m\frac{\partial}{\partial t}\left(\frac{v}{\sqrt{1-(\frac{v}{c})^2}}\right) \quad \text{(Einstein)} \tag{31}$$

The question is: When does the relativistic equation of motion effectively look like Newton's equation? Answer: when the γ-factor is approximately unity and plays no big role, as we see by comparing these two equations with each other! Hence, the worst case is the largest velocity (at Alpha Centauri), so by (6) we find

$$\boxed{\gamma_{\text{AlphaCentauri}} = 1/\sqrt{1-(0.1)^2} = 1.005 \approx 1,} \; [\mathbf{0.5}\ \text{Point(s)}] \tag{32}$$

so even when the velocity is one tenth of c, the γ-factor differs only five thousandth from 1, which again makes the relativistic motion behave much like classical motion.

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e) (REMARK FOR TEACHING ASSISTANTS: Suggested rating is [**0.5** Point(s)] for the distance plot, [**0.5** Point(s)] for the velocity plot — as long as the asymptotic behavior of uniform motion is visible — and [**1** Point(s)] for the comparison with the previous subexercise). The numerical prefactor in (23) is $1.99 \times 10^{21}\,\frac{\text{m}^3}{\text{s}^2} = 586.3\,\frac{\text{AU}^3}{\text{y}^2}$. Using the former value in the numerical solution yields Fig. 3 (which is an acceptable solution), while the latter value yields Fig. 4 (which should be presented in the problem session). As

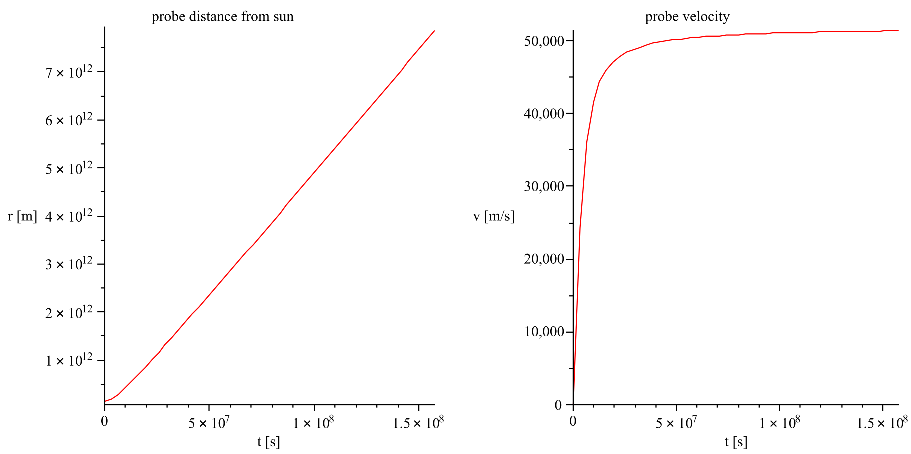

Figure 3: Distance $r(t)$ from sun and velocity $v(t)$ of the space probe in SI units for 5 years.

Fig. 4 shows, the probe needs about 23.3 y now to reach the edge of the solar system (250 AU) in this more realistic model, more than 8 times as long as in the approximation of the previous subexercise. Since the velocity saturates pretty quickly at about $10.9\,\frac{\text{AU}}{\text{y}}$, the time needed to head to Alpha Centauri with this propelling method is circa $\frac{2.8\times10^5\,\text{AU}}{10.9\,\text{AU/y}} \approx 25,700\,\text{y}$, which is way more than in the crude approximation of the previous subexercise.

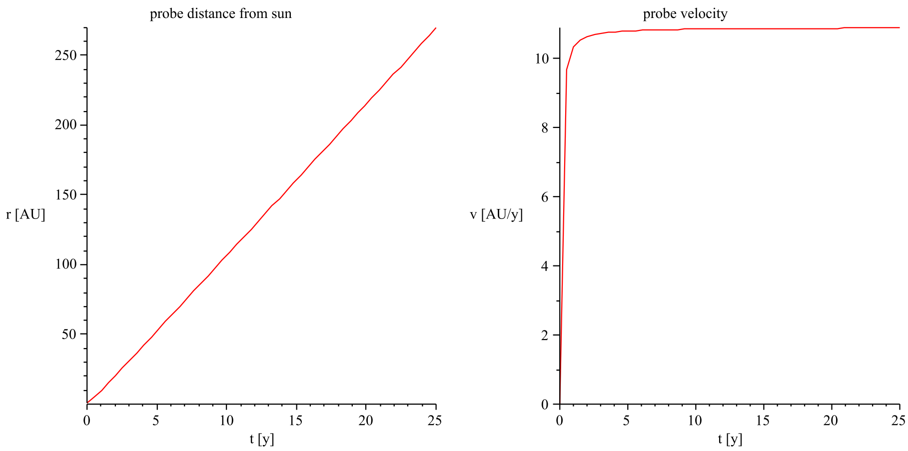

Figure 4: Distance $r(t)$ from sun and velocity $v(t)$ of the space probe in more adequate units AU and years for 25 years.

2 Lorentz Invariants Of The EM Field

Let $\mathcal{K}$ be the rest frame in which the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ were measured. The same measurements are now performed in a frame $\mathcal{K}'$ moving at velocity $\mathbf{v}$ relative to $\mathcal{K}$. This yields the values

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \qquad \mathbf{E}'_{\perp} = \frac{\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B})}{\sqrt{1-(v/c)^2}}, \tag{33}$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \qquad \mathbf{B}'_{\perp} = \frac{\mathbf{B}_{\perp} - (\mathbf{v} \times \mathbf{E})/c^2}{\sqrt{1-(v/c)^2}}, \tag{34}$$

where the indices $\parallel$ and $\perp$ denote the parts of $\mathbf{E}$ and $\mathbf{B}$ parallel and perpendicular to the relative velocity $\mathbf{v}$, respectively. Justify Your calculations properly in the following.

a) Show that the quantity $\mathbf{E} \cdot \mathbf{B}$ is relativistically invariant, i.e., it has the same value as $\mathbf{E}' \cdot \mathbf{B}'$. [**3** Point(s)]

b) Show that the quantity $\mathbf{E}^2 - c^2\mathbf{B}^2$ is relativistically invariant. [**3** Point(s)]

c) Find expressions for $\mathbf{E}'$ and $\mathbf{B}'$ in the low velocity limit, i.e., when $v/c \ll 1$. Compare Your result with the Lorentz force acting on a moving point charge. What is Your conclusion about the physical origin of the Lorentz force? [**3** Point(s)]

Hint: You may need the following vector identities:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{35}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{36}$$

Note: It can be shown, that Maxwell's Equations only permit the two invariants discussed above.

(Remark for teaching assistants: This is a nice exercise to practice the application of vector algebra. No rating suggestions are given here, follow Your instincts or use the Force.

However, the students should put great care in the proper arguments, why some vector products vanish and others do not. Thus, points may be subtracted for sloppy presentation and argumentation. Many students will most likely not use vector arrows and thus make mistakes like switching from vector algebra to scalar algebra, or write down expressions that make no sense, e.g. triple products of vectorial quantities — without arrows — with no parentheses or multiplication signs. Note here, that the cross product is *not* associative, so parentheses are mandatory then. Furthermore, the cross product is anticommutative, while the scalar product is commutative.)

Any arbitrary vector $\mathbf{a}$ can be written as the sum of the parts $\mathbf{a}_{\parallel}$ parallel and $\mathbf{a}_{\perp}$ perpendicular to the velocity $\mathbf{v}$ ($\mathbf{b}$ is another arbitrary vector, not the magnetic field here), where

$$\mathbf{a} \times \mathbf{a} = 0 \tag{37}$$
$$\mathbf{a} \cdot \mathbf{a} = a^2, \tag{38}$$
$$\mathbf{a} = \mathbf{a}_{\perp} + \mathbf{a}_{\parallel}, \tag{39}$$
$$\mathbf{a}_{\parallel} \times \mathbf{v} = 0, \tag{40}$$
$$\mathbf{a}_{\perp} \cdot \mathbf{v} = 0, \tag{41}$$
$$\mathbf{a}_{\perp} \cdot \mathbf{b}_{\parallel} = 0, \qquad \text{(where } \mathbf{b} = \mathbf{b}_{\perp} + \mathbf{b}_{\parallel}\text{)} \tag{42}$$
$$|\mathbf{a} \times \mathbf{v}| = |\mathbf{a}||\mathbf{v}| \sin(\angle(\mathbf{a}, \mathbf{v})). \tag{43}$$

However, note that in general we have

$$\mathbf{a}_{\perp} \times \mathbf{b}_{\perp} \neq 0, \tag{44}$$

since $\mathbf{a}_{\perp}$ and $\mathbf{b}_{\perp}$ only lie in the plane perpendicular to v, but are in general not parallel to each other! Terms involving these products will cancel out in the following.

This is basic stuff, which the students should know already. From these basic relations, we need to derive some more identities for the following computations.

First identity is

$$\mathbf{v} \times \mathbf{a} = \mathbf{v} \times (\mathbf{a}_{\parallel} + \mathbf{a}_{\perp}) \qquad \text{by (39)} \tag{45}$$
$$= \underbrace{\mathbf{v} \times \mathbf{a}_{\parallel}}_{=0 \text{ by } (40)} + \mathbf{v} \times \mathbf{a}_{\perp} \qquad \text{factored out} \tag{46}$$
$$= \mathbf{v} \times \mathbf{a}_{\perp}. \tag{47}$$

Using this identity, we can derive the second identity (note the two different vectors $\mathbf{a}$ and $\mathbf{b}$ here) as

$$\mathbf{b}_{\parallel} \cdot (\mathbf{v} \times \mathbf{a}) = \mathbf{b}_{\parallel} \cdot (\mathbf{v} \times \mathbf{a}_{\perp}) \qquad \text{by (47)} \tag{48}$$
$$= \mathbf{v} \cdot \underbrace{(\mathbf{a}_{\perp} \times \mathbf{b}_{\parallel})}_{\perp \mathbf{v}} \tag{49}$$
$$= 0. \qquad \text{by (42)} \tag{50}$$

Furthermore we need these products with the same vector $\mathbf{a}$ as

$$\mathbf{a}_\perp \cdot (\mathbf{v} \times \mathbf{a}) = \mathbf{a}_\perp \cdot (\mathbf{v} \times \mathbf{a}_\perp) \qquad \text{by (47)} \tag{51}$$

$$= \mathbf{v} \cdot \underbrace{(\mathbf{a}_\perp \times \mathbf{a}_\perp)}_{=0 \text{ by } (37)} \tag{52}$$

$$= 0. \qquad \text{by (42)} \tag{53}$$

Finally, from the the first identity we also find the third identity, where $\alpha = \pi/2$ is the angle between $\mathbf{v}$ and $\mathbf{a}_\perp$:

$$(\mathbf{v} \times \mathbf{a})^2 = (\mathbf{v} \times \mathbf{a}_\perp)^2 \qquad \text{by (47)} \tag{54}$$

$$= |\mathbf{v}|^2 |\mathbf{a}_\perp|^2 \underbrace{\sin^2(\alpha)}_{=1 \text{ by remark}} \qquad \text{by (43)} \tag{55}$$

$$=: v^2 \mathbf{a}_\perp^2. \tag{56}$$

(Remark for teaching assistants: **At this point, it may be necessary to stress that quantities without vector arrows/bold letters are always meant to be the absolute values/magnitudes of said vectors. Not everyone may know this custom.**)

Furthermore, we need the γ-factor again:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \text{with } \beta = v/c. \tag{57}$$

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a) We can split the field vectors as follows up into

$$\mathbf{E}' \cdot \mathbf{B}' = \left(\mathbf{E}'_\parallel + \mathbf{E}'_\perp\right) \cdot \left(\mathbf{B}'_\parallel + \mathbf{B}'_\perp\right) \qquad \text{by (39)} \tag{58}$$

$$= \mathbf{E}'_\parallel \mathbf{B}'_\parallel + \mathbf{E}'_\perp \mathbf{B}'_\perp \qquad \text{by (42)} \tag{59}$$

$$= \underbrace{\mathbf{E}_\parallel \cdot \mathbf{B}_\parallel}_{\text{term 1}} + \gamma^2 \underbrace{\left(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}\right)}_{\text{term 2}} \underbrace{\left(\mathbf{B}_\perp - \frac{\mathbf{v} \times \mathbf{E}}{c^2}\right)}_{\text{term 3}} \qquad \text{by (33), (34)} \tag{60}$$

We factor out the product of terms 2 and 3 (without γ^2), giving

$$\text{term } 2 \cdot 3 = \mathbf{E}_\perp \cdot \mathbf{B}_\perp + \underbrace{(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{B}_\perp}_{=0 \text{ by } (53)} \tag{61}$$

$$- \frac{1}{c^2} \underbrace{\mathbf{E}_\perp \cdot (\mathbf{v} \times \mathbf{E})}_{=0 \text{ by } (53)} - \frac{1}{c^2} (\mathbf{v} \times \mathbf{B}) \cdot (\mathbf{v} \times \mathbf{E}) \tag{62}$$

$$= \underbrace{\mathbf{E}_\perp \cdot \mathbf{B}_\perp}_{\text{term 4}} \underbrace{- \frac{1}{c^2} (\mathbf{v} \times \mathbf{B}) \cdot (\mathbf{v} \times \mathbf{E})}_{\text{term 5}}. \tag{63}$$

Now, let's look at term 5:

$$-\frac{1}{c^2}(\mathbf{v}\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{E}) = -\frac{1}{c^2}(\mathbf{v}\times\mathbf{B}_\perp)\cdot(\mathbf{v}\times\mathbf{E}_\perp) \qquad \text{by (47)} \tag{64}$$

$$= -\frac{1}{c^2}(\mathbf{B}_\perp\times[\mathbf{v}\times\mathbf{E}_\perp])\cdot\mathbf{v} \qquad \text{by (35)} \tag{65}$$

$$= -\frac{1}{c^2}(\mathbf{v}\underbrace{[\mathbf{B}_\perp\cdot\mathbf{E}_\perp]}_{\text{scalar}} - \mathbf{E}_\perp\underbrace{[\mathbf{B}_\perp\cdot\mathbf{v}]}_{=0 \text{ by (41)}})\cdot\mathbf{v} \qquad \text{by (36)} \tag{66}$$

$$= -\frac{v^2}{c^2}\mathbf{B}_\perp\cdot\mathbf{E}_\perp. \qquad \text{by (38)} \tag{67}$$

Now, plugging (63) and (67) into (60) yields

$$\mathbf{E}'\cdot\mathbf{B}' = \mathbf{E}_\parallel\cdot\mathbf{B}_\parallel + \gamma^2(\mathbf{E}_\perp\cdot\mathbf{B}_\perp - \frac{v^2}{c^2}\mathbf{B}_\perp\cdot\mathbf{E}_\perp) \tag{68}$$

$$= \mathbf{E}_\parallel\cdot\mathbf{B}_\parallel + \underbrace{\gamma^2(1-\frac{v^2}{c^2})}_{=1 \text{ by (57)}}\mathbf{E}_\perp\cdot\mathbf{B}_\perp \qquad \text{dot product is commutative} \tag{69}$$

$$= \mathbf{E}_\parallel\cdot\mathbf{B}_\parallel + \mathbf{E}_\perp\cdot\mathbf{B}_\perp \tag{70}$$

$$= \mathbf{E}\cdot\mathbf{B}. \qquad \text{by (39)} \tag{71}$$

Thus, this dot product is relativistically invariant. QED

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b) We start again with the splitting in parallel and perpendicular parts:

$$(\mathbf{E}')^2 - c^2(\mathbf{B}')^2 = (\mathbf{E}'_\parallel + \mathbf{E}'_\perp)^2 - c^2(\mathbf{B}'_\parallel + \mathbf{B}'_\perp)^2 \qquad \text{by (39)} \tag{72}$$

$$= (\mathbf{E}'_\parallel)^2 + 2\underbrace{\mathbf{E}'_\parallel\cdot\mathbf{E}'_\perp}_{=0 \text{ by (42)}} + (\mathbf{E}'_\perp)^2 - c^2[(\mathbf{B}'_\parallel)^2 + 2\underbrace{\mathbf{B}'_\parallel\cdot\mathbf{B}'_\perp}_{=0 \text{ by (42)}} + (\mathbf{B}'_\perp)^2] \tag{73}$$

$$= \underbrace{(\mathbf{E}'_\parallel)^2 + (\mathbf{E}'_\perp)^2}_{\text{term 1}} - c^2\underbrace{[(\mathbf{B}'_\parallel)^2 + (\mathbf{B}'_\perp)^2]}_{\text{term 2}} \tag{74}$$

Let's treat the electric part in term 1 first, giving

$$(\mathbf{E}'_\parallel)^2 + (\mathbf{E}'_\perp)^2 = \mathbf{E}_\parallel^2 + \gamma^2(\mathbf{E}_\perp + [\mathbf{v}\times\mathbf{B}])^2 \qquad \text{by (33)} \tag{75}$$

$$= \mathbf{E}_\parallel^2 + \gamma^2(\mathbf{E}_\perp^2 + 2\underbrace{\mathbf{E}_\perp\cdot[\mathbf{v}\times\mathbf{B}]}_{\neq 0} + \underbrace{[\mathbf{v}\times\mathbf{B}]^2}_{=v^2\mathbf{B}_\perp^2 \text{ by (56)}}) \qquad \text{factored out} \tag{76}$$

$$= \mathbf{E}_\parallel^2 + \gamma^2(\mathbf{E}_\perp^2 + 2\mathbf{E}_\perp\cdot[\mathbf{v}\times\mathbf{B}_\perp] + v^2\mathbf{B}_\perp^2). \qquad \text{(by (47))} \tag{77}$$

The very same identities apply to the magnetic field term 2, but the cross pruduct has a different order *and* a different sign:

$$(\mathbf{B}'_\parallel)^2 + (\mathbf{B}'_\perp)^2 = \mathbf{B}_\parallel^2 + \gamma^2(\mathbf{B}_\perp - \frac{1}{c^2}[\mathbf{v}\times\mathbf{E}])^2 \qquad \text{by (34)} \tag{78}$$

$$= \mathbf{B}_\parallel^2 + \gamma^2(\mathbf{B}_\perp^2 - \frac{2}{c^2}\underbrace{\mathbf{B}_\perp\cdot[\mathbf{v}\times\mathbf{E}]}_{\neq 0} - \frac{1}{c^4}\underbrace{[\mathbf{v}\times\mathbf{E}]^2}_{=v^2\mathbf{E}_\perp^2 \text{ by (56)}}) \qquad \text{factored out} \tag{79}$$

$$= \mathbf{B}_\parallel^2 + \gamma^2(\mathbf{B}_\perp^2 - \frac{2}{c^2}\mathbf{B}_\perp\cdot[\mathbf{v}\times\mathbf{E}_\perp] + \frac{v^2}{c^4}\mathbf{E}_\perp^2). \qquad \text{(by (47))} \tag{80}$$

Below, we will get a term like the following one, which is 0:

$$\mathbf{E}_\perp \cdot [\mathbf{v} \times \mathbf{B}_\perp] + \underbrace{\mathbf{B}_\perp \cdot [\mathbf{v} \times \mathbf{E}_\perp]}_{=\mathbf{E}_\perp \cdot [\mathbf{B}_\perp \times \mathbf{v}] \text{ by } (35)} = \mathbf{E}_\perp \cdot [\mathbf{v} \times \mathbf{B}_\perp] + \mathbf{E}_\perp \cdot \underbrace{[\mathbf{B}_\perp \times \mathbf{v}]}_{=-[\mathbf{v} \times \mathbf{B}_\perp]} \tag{81}$$

$$= 0. \tag{82}$$

Plugging the partial results for term 2 in (80) and term 1 in (77) into (74) yields

$$(\mathbf{E}')^2 - c^2(\mathbf{B}')^2 = (\mathbf{E}'_\parallel)^2 + (\mathbf{E}'_\perp)^2 - c^2[(\mathbf{B}'_\parallel)^2 + (\mathbf{B}'_\perp)^2] \quad \text{by (74)} \tag{83}$$

$$= \mathbf{E}_\parallel^2 + \gamma^2(\mathbf{E}_\perp^2 + 2\mathbf{E}_\perp \cdot [\mathbf{v} \times \mathbf{B}_\perp] + v^2 \mathbf{B}_\perp^2)$$
$$- c^2 \mathbf{B}_\parallel^2 - \gamma^2(c^2 \mathbf{B}_\perp^2 - 2\mathbf{B}_\perp \cdot [\mathbf{v} \times \mathbf{E}_\perp] + \frac{v^2}{c^2}\mathbf{E}_\perp^2) \quad \text{by (77), (80)} \tag{84}$$

$$= \mathbf{E}_\parallel^2 + \underbrace{\gamma^2(1 - \frac{v^2}{c^2})}_{=1 \text{ by } (57)} \mathbf{E}_\perp^2 - c^2[\mathbf{B}_\parallel^2 + \underbrace{\gamma^2(1 - \frac{v^2}{c^2})}_{=1 \text{ by } (57)} \mathbf{B}_\perp^2]$$
$$+ 2\gamma^2 \underbrace{(\mathbf{E}_\perp \cdot [\mathbf{v} \times \mathbf{B}_\perp] + \mathbf{B}_\perp \cdot [\mathbf{v} \times \mathbf{E}_\perp])}_{=0 \text{ by } (82)} \quad \text{rearranged terms} \tag{85}$$

$$= \mathbf{E}_\parallel^2 + \mathbf{E}_\perp^2 - c^2(\mathbf{B}_\parallel^2 + \mathbf{B}_\perp^2) \tag{86}$$

$$= \mathbf{E}^2 - c^2 \mathbf{B}^2. \quad \text{(74) backwards} \tag{87}$$

Thus, this quantity is relativistically invariant. QED

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c) Expanding γ for small velocities, we have

$$\beta = \frac{v}{c} \ll 1 \Rightarrow \gamma = \sqrt{1 - \beta^2} = 1 + \frac{1}{2}\beta^2 + O(\beta^3), \tag{88}$$

so to leading order in the β-factor we get

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}\right) \quad \text{by (33)} \tag{89}$$

$$= \mathbf{E}_\perp + \mathbf{v} \times \mathbf{B} + O(\beta^2), \quad \text{by (88)} \tag{90}$$

$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v} \times \mathbf{E}}{c^2}\right) \quad \text{by (34)} \tag{91}$$

$$= \mathbf{B}_\perp - \frac{\mathbf{v} \times \mathbf{E}}{c^2} + O(\beta^2) \quad \text{by (88)} \tag{92}$$

$$= \mathbf{B}_\perp + O(\beta), \quad \text{by (88)} \tag{93}$$

Thus, the desired low velocity expressions are

$$\mathbf{E}' = \boxed{\mathbf{E} + \mathbf{v} \times \mathbf{B}} \text{[1 Point(s)]} \tag{94}$$

$$\mathbf{B}' = \boxed{\mathbf{B}} \text{[1 Point(s)]} \tag{95}$$

but only $\mathbf{E}'$ is of concern here. The quantities discussed now are shown in Fig. 5. In

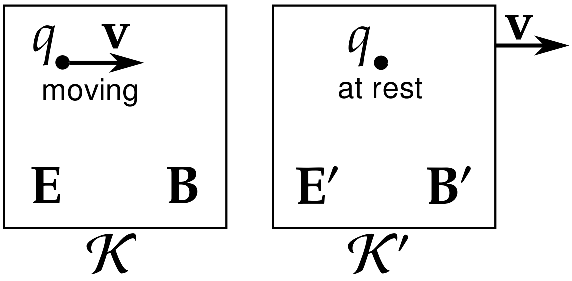

Figure 5: Left: A point charge q moving with velocity $\mathbf{v}$ through the fields $\mathbf{E}$ and $\mathbf{B}$ measured in the rest frame $\mathcal{K}$. Right: In the frame $\mathcal{K}'$ moving with $\mathbf{v}$ along the point charge, the charge is at rest feeling *only* the electric field $\mathbf{E}'$, though the magnetic field $\mathbf{B}'$ is present.

the frame $\mathcal{K}'$ that moves along the point charge with velocity $\mathbf{v}$, the charge is at rest and feels only an electric field

$$\mathbf{F} = q\mathbf{E}'. \tag{96}$$

This is the definition of the electric field, namely the force felt by a test charge at rest (i.e., it experiences an electrostatic force). This field $\mathbf{E}'$, however, is now given by the electric and magnetic field $\mathbf{E}$ and $\mathbf{B}$ measured in $\mathcal{K}$, in which the point charge moves with $\mathbf{v}$. It is given by the Lorentz-transformed fields in the low velocity limit (94) as

$$\begin{aligned}
\mathbf{F} &= q\mathbf{E}' && && (97)\\
&= q(\mathbf{E}'_\parallel + \mathbf{E}'_\perp) && && (98)\\
&= q\left(\mathbf{E}_\parallel + \gamma[\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}]\right) && \text{by (33)} && (99)\\
&\approx q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right). && \text{for low velocities by (94)} && (100)
\end{aligned}$$

which is the conventional Lorentz force.

Thus, the Lorentz force is the manifestation for low velocities of a more complicated force (**namely the Minkowski force**) from the *relativistic* transform behavior of the EM-field. [1 Point(s)]

— Hand in solutions in tutorial on 30.04.2012 —