Karlsruher Institut für Technologie (KIT) SS2012
Institut für theoretische Festkörperphysik
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Tutorial:

☐ **Group 1,**

☐ **Group 2,**

☐ **Group 3.**

Name:____

Problem set 1 for the course "Theoretical Optics"

1 Space Probe Propulsion By Solar Sails

We idealize a solar sail of a space probe by an ideal planar mirror (i.e. no losses due to absorption/heat generation) of rest mass m (Fig. 1). Let W_{i} be the total amount of energy of an incoming localized wave packet at normal incidence to the plane mirror. The wave packet shall be reflected by the mirror and have a total amount of energy W_{r} after the reflection.

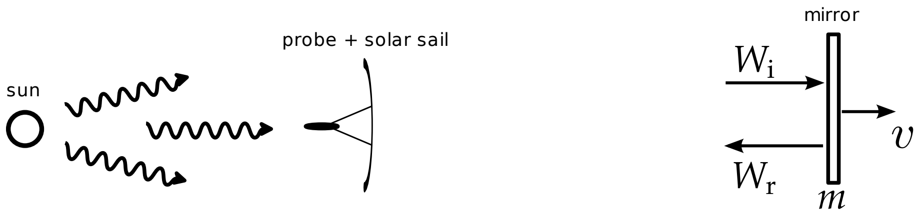

Figure 1: Realistic and idealized model for a solar sail propelled space probe.

a) Assume that the mirror is at rest before the reflection process. Using the relativistic formulae for energy and momentum, write down explicitly the expressions for the conservation of momentum and of energy for the reflection process, respectively. Then, show that the velocity v of the mirror and the energy W_{r} of the reflected wave packet can be written as

$$v = c\frac{(1+2B)^2-1}{(1+2B)^2+1}, \qquad W_{\mathrm{r}} = W_{\mathrm{i}}\frac{1}{1+2B}, \tag{1}$$

and determine the constant B in terms of m and W_{i}. [**4** Point(s)]

b) Find expressions for W_{r} and v in the limits $W_{\mathrm{i}} \ll mc^2$ and $W_{\mathrm{i}} \gg mc^2$: Identify the parameter ϵ which is much smaller than 1 in these cases and apply a Taylor series expansion up to the leading order in ϵ (i.e., beyond the $0th$ order). [**3** Point(s)]

Our sun emits an instantaneous power (luminosity) of $P_{\mathrm{sun}} = \frac{dW}{dt} \approx 3.8 \times 10^{26}\,\mathrm{J/s}$. Now consider an actual space probe with a sail of area A and assume that the weight of the probe is solely determined by its sail, which consists of a material with density $1\,\mathrm{g/m^2}$.

c) First, determine which of the limits (if any) discussed in (b) is appropriate if the probe is launched at the earth (distance r to the sun $= 1\,\mathrm{AU} \approx 1.5 \times 10^{11}\,\mathrm{m}$). Use the appropriate limit to estimate the pressure on the sail as a function of the distance from the sun. Show that this pressure is given by an expression of the form

$$\frac{C}{r^2} \tag{2}$$

and determine the constant C. [**2** Point(s)]

d) With the pressure from (c), first write down the mechanical equation of motion of the probe. Since this equation can not be solved in closed form, make the crude approximation that the pressure remains constant at the value it has for $x = 1\,\mathrm{AU}$. Further assume a sail with area $A = 100 \times 100\,\mathrm{m}$. After which time would the probe leave our solar system ($\approx 250\,\mathrm{AU}$) and when could it reach Alpha Centauri ($2.8 \times 10^5\,\mathrm{AU}$)? Which velocities does the probe have then? Is the non-relativistic treatment still valid when the probe reached the respective distances? [**2** Point(s)]

e) Use a numerical method (e.g., the ordinary differential equation solvers built into Maple, Mathematica, Matlab, etc.) to solve the full differential equation from (d). Assume that the probe is launched from earth and is in rest when the sail is expanded. Plot the distance of the probe and its velocity as a function of time and compare with the result from (d). [**2 BONUS** Point(s)]

2 Lorentz Invariants Of The EM Field

Let $\mathcal{K}$ be the rest frame in which the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ were measured. The same measurements are now performed in a frame $\mathcal{K}'$ moving at velocity $\mathbf{v}$ relative to $\mathcal{K}$. This yields the values

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \qquad \mathbf{E}'_\perp = \frac{\mathbf{E}_\perp + (\mathbf{v} \times \mathbf{B})}{\sqrt{1-(v/c)^2}}, \tag{3}$$

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel, \qquad \mathbf{B}'_\perp = \frac{\mathbf{B}_\perp - (\mathbf{v} \times \mathbf{E})/c^2}{\sqrt{1-(v/c)^2}}, \tag{4}$$

where the indices $\parallel$ and $\perp$ denote the parts of $\mathbf{E}$ and $\mathbf{B}$ parallel and perpendicular to the relative velocity $\mathbf{v}$, respectively. Justify Your calculations properly in the following.

a) Show that the quantity $\mathbf{E} \cdot \mathbf{B}$ is relativistically invariant, i.e., it has the same value as $\mathbf{E}' \cdot \mathbf{B}'$. [**3** Point(s)]

b) Show that the quantity $\mathbf{E}^2 - c^2\mathbf{B}^2$ is relativistically invariant. [**3** Point(s)]

c) Find expressions for $\mathbf{E}'$ and $\mathbf{B}'$ in the low velocity limit, i.e., when $v/c \ll 1$. Compare Your result with the Lorentz force acting on a moving point charge. What is Your conclusion about the physical origin of the Lorentz force? [**3** Point(s)]

Hint: You may need the following vector identities:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{5}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{6}$$

Note: It can be shown, that Maxwell's Equations only permit the two invariants discussed above.

— Hand in solutions in tutorial on 30.04.2012 —