1 Review of Electromagnetism

1.1 Maxwell's equations in Vacuum

Electromagnetism provides most of the basic underlying laws for theoretical optics. Therefore it is very important to give a brief but catchy description of light matter interaction by means of Maxwell's equations and their direct consequences. Conservation laws are deduced and the Maxwell stress tensor is derived.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \text{Coulomb's Law} \tag{1.1}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \qquad \text{Faraday's Law} \tag{1.2}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \text{Gauß' Law for magnetic fields} \tag{1.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E} \qquad \text{Ampere's Law} \tag{1.4}$$

Here, $c^2 = \frac{1}{\varepsilon_0 \mu_0}$ is the square of the vacuum speed of light c, $\mathbf{E}$ the electric field vector, $\mathbf{B}$ the magnetic induction, ρ the electric charge density and $\mathbf{j}$ the electric current density. The electric and magnetic field exert forces on the charge density ρ and the current density $\mathbf{j}$ according to

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \qquad \text{Lorentz Law} \tag{1.5}$$

$\mathbf{f}$ is a force density or more accurately a momentum current density. In simple cases such as a moving charges, the current density equals $\mathbf{j} = \rho \mathbf{v}$ leading to the common form for the Lorentz force density $\mathbf{f} = \rho\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ where $\mathbf{v}$ is the velocity of the charge carriers. In cgs-units, this reads $\mathbf{f} = \rho\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$ and the factor v/c suggests that the second term can be derived from the first term using the principles of special relativity. At first sight that aspect seems not to be of a great importance, but as theoretical optics and nonlinear optics are more and more focusing on so called *slow light effects*, the emphasis of the factor v/c here is important. A brief overview of special relativity is given in the last chapter.

The electromagnetic (EM) field is a dynamical system whose state is completely determined by specifying the functions $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. As a dynamical system, the electromagnetic field contains and transports energy, momentum, angular momentum, etc., which are completely described when $\mathbf{E}$ and $\mathbf{B}$ are specified.

There is a difference between a physical quantity, the state of a system, and the value of a physical quantity in a given state of the system. Relations between physical quantities are independent of any concrete state of a system. These are the natural laws, like Maxwell's equations. In any given state of the system, all physical quantities are associated with a corresponding value (e. g., the values of the momentum density distribution when the electromagnetic field is in the state of a plane wave). In classical physics (including electromagnetism) this means that in a given state certain variables (values, functions, etc.) are specified and physical quantities can be computed by inserting these variables in the corresponding expressions for physical quantities. In quantum mechanics 1. + 2. are true, 3. does not apply anymore (physical quantities may not be given by definite values then, but rather by probabilistic distributions of possible values).

When EM-fields are described by real valued functions, the energy density w of the EM-field equals

$$w(\mathbf{r},t) = \frac{1}{2}\varepsilon_0 \mathbf{E}^2(\mathbf{r},t) + \frac{1}{2\mu_0}\mathbf{B}^2(\mathbf{r},t), \qquad \text{(Energy density)} \tag{1.6}$$

and the associated energy current density, the Poynting vector $\mathbf{S}$, reads as

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t) \qquad \text{(Poynting vector)} \tag{1.7}$$

The source or drain density σ of the energy of the electromagnetic field (gain or dissipation: σ being *not* the conductivity) is

$$\sigma(\mathbf{r},t) = -\mathbf{j}(\mathbf{r},t) \cdot \mathbf{E}(\mathbf{r},t). \tag{1.8}$$

The above quantities are related through (proof via Maxwell's equations)

$$\partial_t w(\mathbf{r},t) + \boldsymbol{\nabla} \cdot \mathbf{S}(\mathbf{r},t) = \sigma(\mathbf{r},t) \qquad \text{(Continuity equation)} \tag{1.9}$$

The equation (1.9) has the general structure of a continuity equation, here it is energy conservation. It says that the energy density inside an infinitesimal volume around $\mathbf{r}$ can change by either energy "streaming in or out" of the volume described by $\mathbf{S}$ or having a source or drain of energy inside the volume (described by σ). If we have no source or drain, then energy is conserved. Electric charge and the electric current density are also related by a continuity equation (proof via Maxwell's equations):

$$\partial_t \rho(\mathbf{r},t) + \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r},t) = 0 \qquad \text{(Current density relation)} \tag{1.10}$$

In other words: charge is conserved, meaning, charge cannot vanish at one position and instantly appear somewhere else in space, which would also conserve charge globally. Instead charge has to move continuously along a path in space, creating a current. What follows is that charge is even conserved locally, expressed by the continuity equation.

Momentum density of the EM-field can be used for applications like the optical tweezer. That means, coherent light is used to track particles only by the influence of electromagnetic forces. The following derivation of Maxwell's stress tensor can also be found in standard text books on electrodynamics, e.g. [1]. We want to know, what momentum the fields carry. We consider a point charge q in an EM-field and consider a small region of space (volume V) that contains this charge (see Fig. 1.1).

$$\frac{\mathrm{d}\mathbf{P}_{\text{mech}}}{\mathrm{d}t} = \mathbf{F}(t) \qquad \text{(Electromagnetic force)} \tag{1.11}$$
$$= \iiint\limits_V \mathbf{f}(\mathbf{r},t)\mathrm{d}V\,,$$

$\mathbf{P}_{\text{mech}}$ is the mechanical momentum carried by the charge. $\mathbf{F}$ is the force acting on the charge by the fields and $\mathbf{f}$ is the force density, i.e., force per volume, see Fig. 1.1.

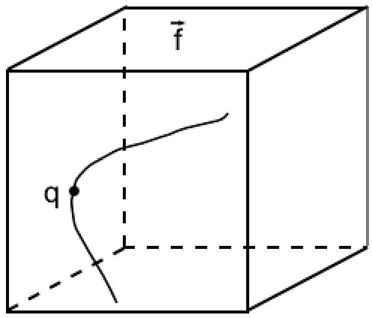


Figure 1.1: Point charge q in an EM-field. The Lorentz force (force density $\mathbf{f}$) acts on the charge and changes its mechanical momentum $\mathbf{P}_{\text{mech}}$.

The external fields exert a force acting on the charge, which moves to a new position. Thus, the charge acquired momentum that flowed from the fields into the charge. The charge causes a field itself, which is associated with this charge. Thus, the charge motion also changed the total field (external field + charge field). So instead of regarding this as an interaction between external fields and charge, we can also look at the problem as an interaction between external fields and field carried by the charge, oppressing q and the associated charge density ρ in the equations. Mathematically, this is facilitated by using Maxwell's equations and the Lorentz force law by

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}. \tag{1.12}$$

$$\rho = \varepsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E} \tag{1.13}$$

$$\mathbf{j} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{B} - \varepsilon_0\partial_t\mathbf{E} \tag{1.14}$$

$$\Rightarrow \mathbf{f} = \varepsilon_0\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{E}\right) + \frac{1}{\mu_0}\left(\boldsymbol{\nabla}\times\mathbf{B}\right)\times\mathbf{B} - \varepsilon_0\left(\partial_t\mathbf{E}\right)\times\mathbf{B} \tag{1.15}$$

We can draw the conclusion that the force exerted on a small volume of charges by external fields is fully determined by the total field configuration, fields due to charges in the volume + external fields. A force is a momentum current (momentum per time) and a force density $\mathbf{f}$ is a momentum current density (momentum per time and volume), similar to a charge current density $\mathbf{j}$ (charge per time and area). We have to reformulate the above expressions to derive the continuity equation for the momentum current density. From these equations above the momentum current density or force density, or more precisely, the stress tensor, associated with the EM-field is derived. The last term of (1.15) reads

$$-(\partial_t \mathbf{E}) \times \mathbf{B} = \mathbf{B} \times \partial_t \mathbf{E} = -\partial_t (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \underbrace{(\partial_t \mathbf{B})}_{=-\boldsymbol{\nabla}\times\mathbf{E}} \tag{1.16}$$

Inserting this into the equation for $\mathbf{f}$ above yields

$$\mathbf{f} = \varepsilon_0 \mathbf{E} (\boldsymbol{\nabla} \cdot \mathbf{E}) - \varepsilon_0 \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) - \frac{1}{\mu_0} \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) - \varepsilon_0 \partial_t (\mathbf{E} \times \mathbf{B}) \tag{1.17}$$

$$= \varepsilon_0 \Big(\mathbf{E} (\boldsymbol{\nabla} \cdot \mathbf{E}) - \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) \Big) + \frac{1}{\mu_0} \Big(\mathbf{B} \underbrace{(\boldsymbol{\nabla} \cdot \mathbf{B})}_{=0} - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) \Big)$$

$$- \partial_t (\varepsilon_0 \mathbf{E} \times \mathbf{B}) \tag{1.18}$$

Substituting that result into the force term (1.11) and bringing all time derivatives to one side yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbf{P}_{\text{mech}} + \iiint_V \varepsilon_0 \mathbf{E} \times \mathbf{B} \mathrm{d}V \right) = \iiint_V \Big(\varepsilon_0 \left(\mathbf{E} (\boldsymbol{\nabla} \cdot \mathbf{E}) - \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) \right) \tag{1.19}$$

$$+ \frac{1}{\mu_0} \left(\mathbf{B} (\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) \right) \Big) \mathrm{d}V \tag{1.20}$$

Since $\mathbf{P}_{\text{mech}}$ is the momentum of the particle, we have that $\iiint_V \varepsilon_0 \mathbf{E} \times \mathbf{B} \mathrm{d}V$ is the momentum of the electromagnetic field. The change of the total combined momentum (fields+mechanical) in the volume V per time is given by the expression on the right, the momentum current associated with the fields. Both terms seperately obey an identity of the following form (proof by calculation)

$$\frac{1}{2} \boldsymbol{\nabla} \left(\mathbf{B}^2 \right) = (\boldsymbol{\nabla} \cdot \mathbf{B}) \mathbf{B} + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) \tag{1.21}$$

Therefore, we can transform the r.h.s. to have the form of a divergence term so that we can apply Gauss' theorem:

$$\mathbf{B} (\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \mathbf{B} (\boldsymbol{\nabla} \cdot \mathbf{B}) + (\mathbf{B} \cdot \boldsymbol{\nabla}) \mathbf{B} - \frac{1}{2} \boldsymbol{\nabla} \left(\mathbf{B}^2 \right)$$

$$=: \boldsymbol{\nabla} \cdot \left(\overleftrightarrow{T} \right) \tag{1.22}$$

The definition of the divergence of a 2nd-rand Tensor T is

$$\left[\boldsymbol{\nabla} \cdot \overleftrightarrow{T} \right]_k = \sum_{i=1}^{3} \partial_{x_i} T_{ik}(\mathbf{r}). \tag{1.23}$$

Finally we define the stress Tensor by applying Gauss' theorem for both $\mathbf{E}$ and $\mathbf{B}$ and using (1.9) for the energy-density we derive

$$
\begin{aligned}
T_{ik} =& \varepsilon_0 E_i E_k + \frac{1}{\mu_0} B_i B_k - w(\mathbf{r},t)\delta_{ik} \\
=& \varepsilon_0 E_i E_k + \frac{1}{\mu_0} B_i B_k - [\frac{1}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2]\delta_{ik}
\end{aligned} \tag{1.24}
$$

By application of the outer (dyadic) product we introduce the Maxwell Stress Tensor

$$
\overleftrightarrow{T} = \varepsilon_0 \mathbf{E} : \mathbf{E} + \frac{1}{\mu_0}\mathbf{B} : \mathbf{B} - \frac{1}{2}\left(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right)\mathbb{1} \tag{1.25}
$$

$$
\tag{1.26}
$$

With this definition, we can write the conservation of total momentum (fields+mechanical) as follows

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathbf{P}_{\mathrm{mech}} + \iiint\limits_V \mathbf{P}_{\mathrm{em}}(\mathbf{r},t)\mathrm{d}V\right\} = \iiint\limits_V \mathrm{div}\overleftrightarrow{T}\mathrm{d}V \tag{1.27}
$$

$$
= \oiint\limits_{\partial V}\left(\overleftrightarrow{T}\cdot\hat{n}\right)\mathrm{d}A \tag{1.28}
$$

The momentum density of the EM-field equals

$$
\mathbf{P}_{\mathrm{em}}(\mathbf{r},t) = \varepsilon_0 \mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) = \frac{1}{c^2}\mathbf{S}(\mathbf{r},t). \tag{1.29}
$$

Finally, we gain the expression for the Lorentz force density

$$
\partial_t \mathbf{P}_{\mathrm{em}}(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\left(-\overleftrightarrow{T}\right) = -\mathbf{f}(\mathbf{r},t) \tag{1.30}
$$

$$
= -\rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) \tag{1.31}
$$

This is a continuity equation for the momentum $\mathbf{P}_{\mathrm{em}}$ associated with the EM-field: $\mathbf{P}_{\mathrm{em}}$ is the momentum density (where the momentum actually is), $\overleftrightarrow{T}$ the momentum current density (where the momentum goes due to the regular dynamics of the fields) and the Lorentz force density is the source or drain density of the EM-field's momentum, i.e., the fields loose momentum when charges or currents have to be accelerated/generated.

The components T_{ik} of the stress tensor have the following meaning: It is the force per unit area in direction $\hat{\mathbf{e}}_i$ acting on the surface being normal in direction $\hat{\mathbf{e}}_k$. Thus, T_{ii} are pressures (forces normal to surfaces), whereas $T_{ik}, i \neq k$ are shears (forces parallel to surfaces). Note that only the total fields enter in the equations, which contain all information about the charge distributions on which the forces act.

The integral of all these force densities over a finite surface ∂V is the total net force acting on the volume, causing it to spin/get deformed or be pushed/ pulled into some direction. This effect is exploited technically by optical tweezers, which can hold microscopic objects (e.g., organic cells) by pure light force from a laser.

The EM-field also contains angular momentum. The corresponding angular momentum density with respect to an arbitrarily chosen origin is given as usual in terms of the (EM-field's) momentum density as

$$\mathbf{l}(\mathbf{r},t) = \mathbf{r} \times \mathbf{p}(\mathbf{r},t). \tag{1.32}$$

1.2 Maxwell's equations in Matter

The so-called microscopic Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \text{Coulomb's Law} \tag{1.33}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \qquad \text{Faraday's Law} \tag{1.34}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Gauß' Law for magnetic fields} \tag{1.35}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\partial_t \mathbf{E} \qquad \text{Ampere's Law} \tag{1.36}$$

If we consider EM-fields in matter, it is useful to separate charges and currents into external contributions and those that are provided by or contained in the matter.

Note: "External" does not mean "outside of matter" but rather "controllable from the outside". The changes in charge and current densities in matter depend in very complicated and usually unknown ways on the fields because they interact with each other.

$$\rho = \rho_{\text{ext}} + \rho_{\text{mat}} \tag{1.37}$$

$$\mathbf{j} = \mathbf{j}_{\text{ext}} + \mathbf{j}_{\text{mat}} \tag{1.38}$$

From our experience, we know that the complicated variations of ρ_{mat} and $\mathbf{j}_{\text{mat}}$ on atomic scales are not resolved on the much larger scales of optical wavelengths. This suggests that we do not need to consider the local (**on atomic level**) fields as described by the **microscopic** Maxwell's equations above. In contrary fields that are averaged, and therefore smoothed, over atomic distances, actually volumes V_0, are commonly used. These are the so-called macroscopic fields and lead to the so-called macroscopic Maxwell's equations (see Fig. 1.2).

$$\mathbf{E}(\mathbf{r},t) \mapsto \langle \mathbf{E}(\mathbf{r},t) \rangle := \frac{1}{V_0} \int\limits_{V_0} \mathrm{d}^3 r' \; \mathbf{E}\left(\mathbf{r} - \mathbf{r}', t\right). \tag{1.39}$$

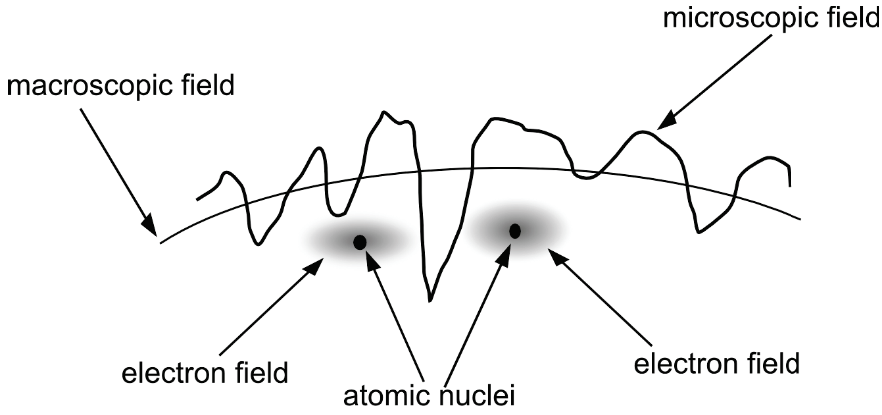


Figure 1.2: Microscopic and macroscopic electric field in matter

Then, it is natural to further split ρ_{mat} and $\mathbf{j}_{\text{mat}}$ into free (i.e. highly mobile beyond the atomic scale) contributions and bound (to the atoms or molecules, i.e., mobile only on the atomic scale) contributions and to associate new fields (polarisation $\mathbf{P}$, magnetisation $\mathbf{M}$) with the bound contributions:

$$\rho_{\text{mat}} = \rho_{\text{free}} + \underbrace{\rho_{\text{pol}}}_{=-\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{r},t)} \tag{1.40}$$

$$\mathbf{j}_{\text{mat}} = \mathbf{j}_{\text{free}} + \underbrace{\mathbf{j}_{\text{pol}}}_{=\partial_t\mathbf{P}(\mathbf{r},t)} + \underbrace{\mathbf{j}_{\text{mag}}}_{=\boldsymbol{\nabla}\times\mathbf{M}(\mathbf{r},t)} \tag{1.41}$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{1.42}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{1.43}$$

$\mathbf{M}$ is the magnetisation, and $\mathbf{P}$ is the electric polarisation and its time derivative equals $\mathbf{j}_{\text{pol}}$ because of the continuity equation for charge.

This gives the macroscopic Maxwell's equations, the Maxwell's equations in matter:

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{\text{ext}} + \rho_{\text{free}} \tag{1.44}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\partial_t\mathbf{B} \tag{1.45}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{1.46}$$

$$\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{j}_{\text{ext}} + \mathbf{j}_{\text{free}} + \partial_t\mathbf{D} \tag{1.47}$$

The above equations contain the averaged fields and not the true local fields. Thus Maxwell's equations in matter are generalizations to the microscopic Maxwell's equations. Thus, through the averaging process, the electromagnetic field and matter become intertwined and it becomes difficult to separate the two subsystems and the physical quantities associated with them. This is most easily seen through the fact that the macroscopic Maxwell's Equations are invariant under the transformation. Proof: We assume an arbitrary vector field N

$$\nabla \cdot (\boldsymbol{\nabla} \times \mathbf{N}) = 0 \tag{1.48}$$

$$\boldsymbol{\nabla} \times \partial_t \mathbf{N} = \partial_t (\boldsymbol{\nabla} \times \mathbf{N}) \tag{1.49}$$

$$\left.\begin{array}{lcl} \mathbf{P} & \mapsto & \mathbf{P} + \boldsymbol{\nabla} \times \mathbf{N} \\ \mathbf{M} & \mapsto & \mathbf{M} - \partial_t \mathbf{N} \end{array}\right\} \Rightarrow \left\{\begin{array}{lcll} \mathbf{E} & \mapsto & \mathbf{E}, & \mathbf{D} \mapsto \mathbf{D} + \boldsymbol{\nabla} \times \mathbf{N}, \\ \mathbf{B} & \mapsto & \mathbf{B}, & \mathbf{H} \mapsto \mathbf{H} + \partial_t \mathbf{N}. \end{array}\right. \tag{1.50}$$

That transformation is called gauge transformation of macroscopic Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{D} =& \nabla(\epsilon_0 \mathbf{E} + \mathbf{P} + \boldsymbol{\nabla} \times \mathbf{N}) = \rho_{fee} + \rho_{ext} + 0 && \text{Coulomb's Law} \quad (1.51) \\ =& \nabla \epsilon_0 \mathbf{E} + \nabla \mathbf{P} + \nabla \cdot (\boldsymbol{\nabla} \times \mathbf{N}) && (1.52) \\ \boldsymbol{\nabla} \times \mathbf{E} =& \partial_t \mathbf{B} && \text{Faraday's Law} \quad (1.53) \\ \nabla \cdot \mathbf{B} =& 0 && \text{Gauss' Law} \quad (1.54) \\ \boldsymbol{\nabla} \times (\mathbf{H} + \partial_t \mathbf{N}) =& \mathbf{j}_{ext} + \mathbf{j}_{free} && \text{Ampere's Law} \quad (1.55) \end{aligned}$$

The last equation yields

$$\begin{aligned} \boldsymbol{\nabla} \times (\mathbf{H} + \partial_t \mathbf{N}) =& \mathbf{j}_{ext} + \mathbf{j}_{free} + \partial_t \mathbf{D} + \partial_t \boldsymbol{\nabla} \times \mathbf{N} && (1.56) \\ =& \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \partial_t \mathbf{N} && (1.57) \end{aligned}$$

for visible light the linear approximation is feasible.

In Optics i. e., for frequencies and wavelengths of the order of visible light, linearizations are often used

$$\mathbf{D} \sim \mathbf{E}, \qquad \mathbf{B} \sim \mathbf{H}. \tag{1.58}$$

In this lecture we will restrict ourselves to such linear relations. The consequences of nonlinear relations between the various fields are the subject of Nonlinear Optics (see lecture by Prof. Jürg Leuthold). In particular for time-harmonic fields ($e^{-\mathrm{i}\omega t}$), such linear relations lead to the simple relations

$$\mathbf{D}(\omega) = \varepsilon_0 \epsilon(\omega) \mathbf{E}(\omega), \tag{1.59}$$

$$\mathbf{B} = \mu_0 \mu(\omega) \mathbf{H}(\omega), \tag{1.60}$$

with the following material parameters

$\epsilon(\omega)$:	Dielectric function	(Permittivity)
$\mu(\omega)$:		(Permeability).

In the time-domain, these simple products become the rather complicated convolution integrals

$$\mathbf{D}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \mathrm{d}t' \, \epsilon\,(t-t')\,\mathbf{E}(t'), \tag{1.61}$$

$$\mathbf{B}(t) = \mu_0 \int_{-\infty}^{\infty} \mathrm{d}t' \, \mu\,(t-t')\,\mathbf{H}(t'), \tag{1.62}$$

in which $\epsilon\,(t)$ and $\mu\,(t)$ are the Fourier transforms of $\epsilon(\omega)$ and $\mu(\omega)$.

- named after Hans Kramers (dutch) and Ralph Kronig (german-american)
- very general relation that expresses causality (defined in time-space) in the Fourier transformed frequency-space

• Let us consider the response $X(t,\vec{r})$ of physical system with respect to property $G(t,t';\vec{r},\vec{r}')$ when the system is experiencing an external distortion $f(t',\vec{r}')$. We then have

$$X(t,\vec{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\vec{r}' \int_{-\infty}^{+\infty} \mathrm{d}t' G(t-t';\vec{r}-\vec{r}')f(t',\vec{r}') \tag{1.63}$$

by definition
$X(t,\vec{r})$ is generally called a response function
$G(t-t';\vec{r}-\vec{r}')$ is often called a Green's function
the relation is widely explored for instance in "linear response theory"

assume local behavior in time space, we have $G(t-t';\vec{r}-\vec{r}') = \delta(t-t')G(t-t',\vec{r}-\vec{r}')$ furthermore Causality requires: $G(t-t') = 0$ for $t < t'$, if we assume a local behavior in position space, we have $G(t-t';\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')G(t-t')$. Therefore (we suppress from now on the argument $\vec{r}$, it is not relevant for Kramers-Kronig)

$$X(t) = \int_{-\infty}^{t} \mathrm{d}t' G(t-t')f(t') \tag{1.64}$$

as a particular example, in the lecture we had (in frequency space)

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\epsilon(\omega)\vec{E}(\omega) \tag{1.65}$$

with D Electric displacement field, electric field E and dielectric Polarization P of medium.

Oftenly one assumes, that the polarization depends linearly on the incident field E, such that in frequency space

$$\vec{P}(\omega) = \epsilon_0 \chi_e(\omega) \vec{E}(\omega) \tag{1.66}$$

which defines the electric Susceptibility χ_e and with the first line gives:
$\epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e(\omega) \vec{E}(\omega) = \epsilon_0 \vec{E}\,(1 + \chi_e(\omega)) = \epsilon_0 \epsilon(\omega) \vec{E}(\omega)$ which yields the relation $\epsilon(\omega) = (1 + \chi_e(\omega))$

in time space we have (Fourier-transform of product is this convolution integral)

$$\vec{P}(t) = \int_{-\infty}^{t} \mathrm{d}t' \chi_e(t - t') \vec{E}(t') \tag{1.67}$$

therefore P is the repsonse of the system (medium) caused by an electric field E and the electric susceptibility relating the two must obey causality, i.e. $\chi_e(t - t') = 0$ for $t < t'$

Now using a Fourier-transform on $\chi_e(t - t')$

$$\chi_e(\omega) = \int_{-\infty}^{+\infty} e^{i\omega(t-t')} \chi_e(t - t') \mathrm{d}(t - t') \equiv \int_{-\infty}^{+\infty} e^{i\omega\tau} \chi_e(\tau) \mathrm{d}\tau \tag{1.68}$$

in the last step we used $\tau = t - t'$

open questions:
1.) – how is the causality requirement expressed now in frequency space?
2.) – the result of the Fourier-transformation of a real-valued function $\chi_e(t - t')$ is a complex valued function. Therefore, $\chi_e(\omega) = \mathrm{Re}\,(\chi_e(\omega)) + i\mathrm{Im}\,(\chi_e(\omega))$. Is there a relation between $\mathrm{Re}\,(\chi_e(\omega))$ and $\mathrm{Im}\,(\chi_e(\omega))$?

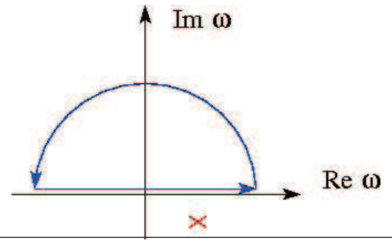
To analyze this function $\chi_e(\omega)$, we continue the function to the complex frequency plane that is we replace $\omega \to \omega_1 = \omega + i\eta$

We wish to consider the closed contour integral

$$\oint \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} \mathrm{d}\omega \tag{1.69}$$

Since the physical quantity $\chi_e(\omega)$ has no poles etc. the integrand has a pole, where $\omega - \omega_0 = 0$, with the above substitution for ω this results in $\omega + i\eta - \omega_0 = 0$, or by rearranging $\omega = \omega_0 - i\eta$, i.e. there is a pole in the lower complex frequency plane as shown by the red cross in the figure.

- the integration path (the contour) is along the blue line in the sketch
- no pole is included



• the radius is extended to infinity, the integration
is splitted into an integral along the real axes
and into an part along the arc of the semi-circle

$$\oint \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega = \int_{-\infty}^{+\infty} \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega + \int_{semi-circle} \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega \tag{1.70}$$

• for an integrable function $\chi_e(\omega)$ we have that the integral along the infinitely large semicircle vanishes
• on the other hand, from Cauchy's integral formula (proven in Mathematics) $\oint \frac{\chi_e(\omega)}{\omega-\omega_0} d\omega = 0$ because there is no pole included in the integration contour
Therefore we have

$$0 = \oint \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega = \int_{-\infty}^{+\infty} \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega + 0 \tag{1.71}$$

$$0 = \int_{-\infty}^{+\infty} \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega \tag{1.72}$$

Furthermore, we use the identity (proof follows in a minute) (understood in the limit of $\eta \to 0$ and when integrated over)

$$\frac{1}{\omega - \omega_0 + i\eta} = P(\frac{1}{\omega - \omega_0}) - i\pi\delta(\omega - \omega_0) \tag{1.73}$$

P is the Cauchy principal value, defined for any function $f(\omega)$ by (limit $\delta \to 0$)

$$P \int_{-\infty}^{+\infty} \frac{f(\omega)}{\omega - \omega_0} := \int_{-\infty}^{\omega_0 - \delta} \frac{f(\omega)}{\omega - \omega_0} + \int_{\omega_0 + \delta}^{+\infty} \frac{f(\omega)}{\omega - \omega_0} \tag{1.74}$$

Therefore using this identy (integration over δ-function is trivially done)

$$0 = \int_{-\infty}^{+\infty} \frac{\chi_e(\omega)}{\omega - \omega_0 + i\eta} d\omega \tag{1.75}$$

$$= P \int_{-\infty}^{+\infty} \frac{\chi_e(\omega)}{\omega - \omega_0} d\omega \; - \; i\pi\chi_e(\omega_0) \tag{1.76}$$

replacing $\omega \to \omega'$ and $\omega_0 \to \omega$ we obtain

$$i\pi\chi_e(\omega) = P \int_{-\infty}^{+\infty} \frac{\chi_e(\omega')}{\omega' - \omega} d\omega' \tag{1.77}$$

now taking the real part on both sides (remember $i * i = -1$)

$$-\pi \mathrm{Im}\chi_e(\omega) = P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\chi_e(\omega')}{\omega' - \omega} d\omega' \tag{1.78}$$

$$\mathrm{Im}\chi_e(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\chi_e(\omega')}{\omega' - \omega} d\omega' \tag{1.79}$$

and taking the imaginary part on both sides

$$\pi \text{Re}\chi_e(\omega) = P\int_{-\infty}^{+\infty} \frac{\text{Im}\chi_e(\omega')}{\omega' - \omega} \text{d}\omega' \tag{1.80}$$

$$\text{Re}\chi_e(\omega) = +\frac{1}{\pi}P\int_{-\infty}^{+\infty} \frac{\text{Im}\chi_e(\omega')}{\omega' - \omega} \text{d}\omega' \tag{1.81}$$

to further simplify the expression, we will now consider a property of the Fourier-transform

a physical quantity in time-space (a response function, which is measurable) is a real-valued function say $f(t)$. This function can always be splitted into two parts
$f(t) = f_{even}(t) + f_{odd}(t)$
where the even part has the property $f_{even}(-t) = f_{even}(+t)$
and the odd part has the property $f_{odd}(-t) = -f_{odd}(+t)$

now looking at the Fourier-transform $f(\omega)$

$$f(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} f(t)\text{d}t \tag{1.82}$$

now using the above split into even and odd part together with the identy
$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ we obtain

$$f(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t}(f_{even}(t) + f_{odd}(t))\text{d}t = \int_{-\infty}^{+\infty} (\cos(\omega t) + i\sin(\omega t))(f_{even}(t) + f_{odd}(t))\text{d}t \tag{1.83}$$

$$= \int_{-\infty}^{+\infty} \cos(\omega t) f_{even}(t)\text{d}t + i\int_{-\infty}^{+\infty} \sin(\omega t) f_{even}(t)\text{d}t \tag{1.84}$$

$$+ \int_{-\infty}^{+\infty} \cos(\omega t) f_{odd}(t)\text{d}t + i\int_{-\infty}^{+\infty} \sin(\omega t) f_{odd}(t)\text{d}t \tag{1.85}$$

we have $\cos(\omega t)$ is an even function and $\sin(\omega t)$ is an odd function of t

furthermore, an integral with symmetric boundaries $\int_{-\infty}^{+\infty}$ yields zero when the integrand is an odd function, because $\int_{-\infty}^{+\infty} g = \int_{-\infty}^{0} g + \int_{0}^{+\infty} g$, both terms differ in sign (think e.g. of the sine-function)

therefore in the above expression (repeated)

$$f(\omega) = \int_{-\infty}^{+\infty} \cos(\omega t) f_{even}(t)\text{d}t + i\int_{-\infty}^{+\infty} \sin(\omega t) f_{even}(t)\text{d}t \tag{1.86}$$

$$+ \int_{-\infty}^{+\infty} \cos(\omega t) f_{odd}(t)\text{d}t + i\int_{-\infty}^{+\infty} \sin(\omega t) f_{odd}(t)\text{d}t \tag{1.87}$$

the second and third term on the right hand side are zero, since an even function times an odd function yields an odd function, odd times odd is even, even times even is even

$$f(\omega) = \int_{-\infty}^{+\infty} \cos(\omega t) f_{even}(t)\mathrm{d}t + i \int_{-\infty}^{+\infty} \sin(\omega t) f_{odd}(t)\mathrm{d}t \tag{1.88}$$

Now, consider the real and the imaginary part of $f(\omega)$

$$\mathrm{Re} f(\omega) = \int_{-\infty}^{+\infty} \cos(\omega t) f_{even}(t)\mathrm{d}t \tag{1.89}$$

$$\mathrm{Im} f(\omega) = \int_{-\infty}^{+\infty} \sin(\omega t) f_{odd}(t)\mathrm{d}t \tag{1.90}$$

Therefore we can state: $\mathrm{Re}f(\omega)$ is an even function of ω (cosine above)
$\mathrm{Im}f(\omega)$ is an odd function of ω (sine above)
Above we had found:

$$\mathrm{Re}\chi_e(\omega) = +\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Im}\chi_e(\omega')}{\omega' - \omega}\mathrm{d}\omega' \tag{1.91}$$

$$\mathrm{Im}\chi_e(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\chi_e(\omega')}{\omega' - \omega}\mathrm{d}\omega' \tag{1.92}$$

multiplying both integrands with $\frac{(\omega'+\omega)}{(\omega'+\omega)}$

$$\mathrm{Re}\chi_e(\omega) = +\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{(\omega'+\omega)\mathrm{Im}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.93}$$

$$\mathrm{Im}\chi_e(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{(\omega'+\omega)\mathrm{Re}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.94}$$

by using $\mathrm{Re}\chi_e(\omega')$ being even and $\mathrm{Im}\chi_e(\omega')$ being odd together with ω' being obviously an odd function of ω' and the observation, that the integration is over symmetric boundaries, we have

$$\mathrm{Re}\chi_e(\omega) = +\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\omega'\mathrm{Im}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.95}$$

$$\mathrm{Im}\chi_e(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\omega\mathrm{Re}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' = -\frac{1}{\pi}\omega P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.96}$$

since the integrands are now even functions of ω' we can substitute $\int_{-\infty}^{+\infty} = 2\int_0^{+\infty}$ yielding

$$\mathrm{Re}\chi_e(\omega) = +\frac{2}{\pi} P \int_{0}^{+\infty} \frac{\omega'\mathrm{Im}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.97}$$

$$\mathrm{Im}\chi_e(\omega) = -\frac{2\omega}{\pi} P \int_{0}^{+\infty} \frac{\mathrm{Re}\chi_e(\omega')}{\omega'^2 - \omega^2}\mathrm{d}\omega' \tag{1.98}$$

Finally we wish to replace the susceptibility $\chi_e(\omega)$ by the dielectric function $\epsilon(\omega)$

In the beginning we had the definition $\epsilon(\omega) = 1 + \chi_e(\omega)$ or $\chi_e(\omega) = \epsilon(\omega) - 1$ with $\mathrm{Re}\chi_e(\omega) = \mathrm{Re}\epsilon(\omega) - 1$ and $\mathrm{Im}\chi_e(\omega) = \mathrm{Im}\epsilon(\omega)$

Using this in the above expressions yields finally

$$\mathrm{Re}\epsilon(\omega) - 1 = +\frac{2}{\pi} P \int_0^{+\infty} \frac{\omega' \mathrm{Im}\epsilon(\omega')}{\omega'^2 - \omega^2} \mathrm{d}\omega' \tag{1.99}$$

$$\mathrm{Im}\epsilon(\omega) = -\frac{2\omega}{\pi} P \int_0^{+\infty} \frac{\mathrm{Re}\epsilon(\omega') - 1}{\omega'^2 - \omega^2} \mathrm{d}\omega' \tag{1.100}$$

Promised proof of:

$$\frac{1}{\omega - \omega_0 + i\eta} = P(\frac{1}{\omega - \omega_0}) - i\pi\delta(\omega - \omega_0) \tag{1.101}$$

$$\frac{1}{\omega - \omega_0 + i\eta} = \frac{1}{\omega - \omega_0 + i\eta} \frac{\omega - \omega_0 - i\eta}{\omega - \omega_0 - i\eta} = \frac{\omega - \omega_0 - i\eta}{(\omega - \omega_0)^2 + \eta^2} = \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \eta^2} - i\eta \frac{1}{(\omega - \omega_0)^2 + \eta^2} \tag{1.102}$$

This means (since ω and ω_0 are real) we have calculated the real and imaginary part

$$\mathrm{Re} \frac{1}{\omega - \omega_0 + i\eta} = \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \eta^2} \tag{1.103}$$

and

$$\mathrm{Im} \frac{1}{\omega - \omega_0 + i\eta} = -\eta \frac{1}{(\omega - \omega_0)^2 + \eta^2} \tag{1.104}$$

As the first step, we wish to consider the imaginary part of $\frac{1}{\omega - \omega_0 + i\eta}$

As said in the beginning, this relation is understood in the limit of $\eta \to 0$ and when integrated over.

Therefore we consider an arbitrary function $f(\omega)$

$$\lim_{\eta \to 0} \int \mathrm{d}\omega \mathrm{Im} \left(\frac{1}{\omega - \omega_0 + i\eta} \right) f(\omega) = -\lim_{\eta \to 0} \int \mathrm{d}\omega \eta \frac{1}{(\omega - \omega_0)^2 + \eta^2} f(\omega) \tag{1.105}$$

first substitution: $y := \omega - \omega_0$ which results in $\frac{\mathrm{d}y}{\mathrm{d}\omega} = 1$ which implies $\mathrm{d}y = \mathrm{d}\omega$ and $\omega = y + \omega_0$

$$= -\lim_{\eta \to 0} \int \mathrm{d}y \frac{\eta}{y^2 + \eta^2} f(y + \omega_0) \tag{1.106}$$

now we employ a second substitution $x := y/\eta$ which results in $\frac{\mathrm{d}x}{\mathrm{d}y} = 1/\eta$ which implies $\mathrm{d}y = \eta \mathrm{d}x$ and $y = \eta x$

$$= \quad -\lim_{\eta\to 0}\int \mathrm{d}x\eta\frac{\eta}{\eta^2x^2+\eta^2}f(\eta x+\omega_0) = -\lim_{\eta\to 0}\int \mathrm{d}x\frac{\eta\eta}{\eta^2x^2+\eta^2}f(\eta x+\omega_0) \tag{1.107}$$

$$= \quad -\lim_{\eta\to 0}\int \mathrm{d}x\frac{1}{x^2+1}f(\eta x+\omega_0) \tag{1.108}$$

now we can interchange the limit and the integration, yielding

$$= -\int \mathrm{d}x\frac{1}{x^2+1}f(\omega_0) = f(\omega_0)\left(-\int \mathrm{d}x\frac{1}{x^2+1}\right) \tag{1.109}$$

now, we remember that

$$-\int \mathrm{d}x\frac{1}{x^2+1} = -\arctan(x)|_{-\infty}^{+\infty} = -(\pi/2-(-\pi/2)) = -\pi \tag{1.110}$$

using this in the above expression

$$= -\pi f(\omega_0) \tag{1.111}$$

Now putting everything together, we have

$$\mathrm{Im}\frac{1}{\omega-\omega_0+i\eta}f(\omega) = -\pi f(\omega_0) \tag{1.112}$$

from this behavior we recognize the delta-function

Because, by definition the delta-function is defined by the action on a test-function according to $\int \mathrm{d}\omega\delta(\omega-\omega_0)f(\omega) := f(\omega_0)$

Therefore, we have proven the equivalence

$$\mathrm{Im}\frac{1}{\omega-\omega_0+i\eta} = -\pi\delta(\omega-\omega_0) \tag{1.113}$$

Which proves the imaginary part of the relation

$$\frac{1}{\omega-\omega_0+i\eta} = P(\frac{1}{\omega-\omega_0}) - i\pi\delta(\omega-\omega_0) \tag{1.114}$$

The proof for the real part follows accordingly.

The Kramers-Kronig relations are a manifestation of causality and are used in experiments to determine $\epsilon(\omega)$.

In matter, it is usually difficult to define the energy density, etc., for the EM-field because the macroscopic Maxwell's equations intertwine the EM-field with material degrees of freedom. However, for a (nearly) time harmonic EM-field described by complex valued functions oscillating in time with $e^{\mathrm{i}\omega t}$, one finds for the energy density and the energy current density

$$\langle w(\omega)\rangle = \frac{1}{4}\varepsilon_0 \,\mathrm{Re}\left(\frac{\mathrm{d}(\omega\epsilon)}{\mathrm{d}\omega}\right)|\mathbf{E}(\omega)|^2 + \frac{1}{4}\,\mathrm{Re}\left(\frac{\mathrm{d}(\omega\mu)}{\mathrm{d}\omega}\right)|\mathbf{H}(\omega)|^2, \tag{1.115}$$

$$\langle \mathbf{S}(\omega)\rangle = \frac{1}{2}\mathrm{Re}\Big(\mathbf{E}(\omega)\times\mathbf{H}^*(\omega)\Big). \tag{1.116}$$

Here, $\langle \quad \rangle$ denotes cycle averaging

$$\langle f\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\; f(t), \tag{1.117}$$

where $f(t) \propto e^{-\mathrm{i}\omega t}$ and $T = \frac{2\pi}{\omega}$.

- The correct form of the Maxwell stress tensor in matter is still a subject of debate. There are at least three different formulations and very recent experiments (from 2009!) appear to suggest that the so-called Abrahams-formulation is correct.
- Remember that the above expressions for the energy density and energy current density are approximate, too. Nearly time harmonic means narrow banded, i.e. the envelope function of the fields varies slowly with respect to the carrier frequency ω. This assumption is the so-called slowly varying envelope approximation.

1.3 Wave Propagation

From the macroscopic Maxwell's equations we derive in the absence of charges and currents and for non-dispersive materials (constant values of ϵ and μ) the wave equation:

$$\Delta\mathbf{E} - \frac{\epsilon\mu}{c^2}\partial_t^2\mathbf{E} = 0, \qquad \text{(d}'\text{Alambert Equation)} \tag{1.118}$$

(1.119)

To work out the main features, we consider the (simplified) 1D scalar case:

$$\Delta U(x,t) - \frac{\epsilon\mu}{c^2}\partial_t^2 U(x,t) = 0 \tag{1.120}$$

In the linear case, we can work with solutions of the type $e^{\mathrm{i}(kx-\omega t)}$ and *have* to take the real part afterwards (because actual physical fields are real). However, k and ω are not independent. Instead, they have to fulfill the dispersion relation

$$k^2 = \frac{\omega^2}{c^2}\epsilon\mu. \tag{1.121}$$

In general, we have dispersive materials $\epsilon \equiv \epsilon(\omega)$ and $\mu \equiv \mu(\omega)$ and then the above relation still holds.

$$\Rightarrow \quad \omega \equiv \omega\,(\mathbf{k}) \qquad \text{Dispersion Relation} \tag{1.122}$$

Very often (i.e. for many materials, or in waveguides and periodic microstructures (Photonic Crystals, Metamaterials) etc.) we have $\omega(k) \neq kc$.

To analyze the consequences, we consider a wave packet (i.e., a superposition of the fundamental solutions described above; remember that we have a linear PDE so that superposition works!)

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k \; A(k) e^{\mathrm{i}(kx - \omega(k)t)} \tag{1.123}$$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; U(x,0) e^{-\mathrm{i}kx}, \tag{1.124}$$

where $A(k)$ is the spatial Fourier Transform of $U(x,0)$.

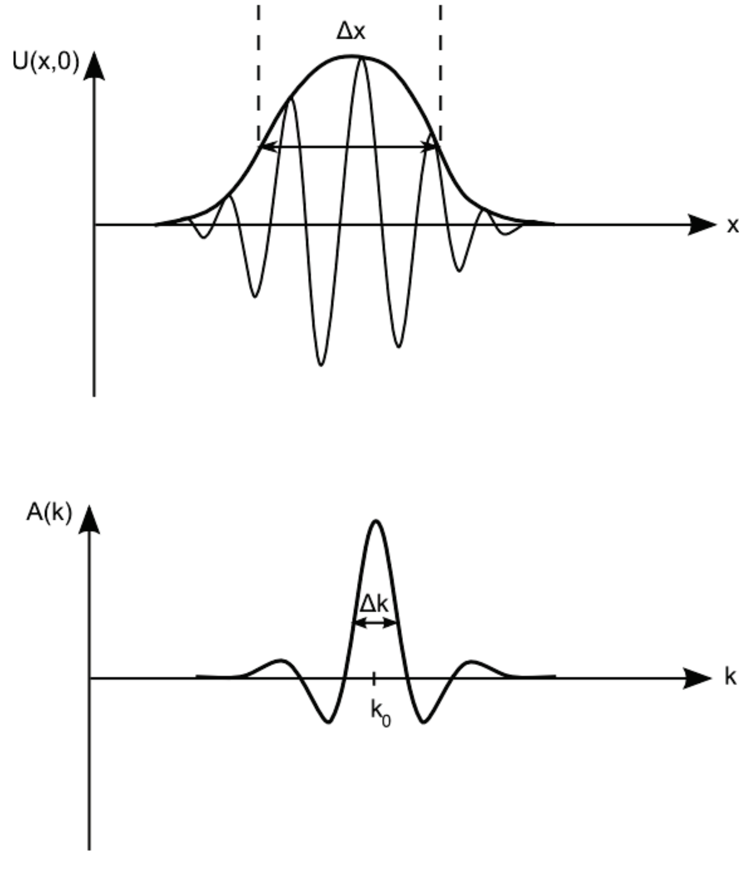


Figure 1.3: A spatially wide pulse $U(x, t = 0)$ has a narrow spectrum $A(k)$ in k-space around some central wavenumber k_0.

A (spatially) wide pulse (which is the regular case in optics)[1] is characterized by a narrow k-spectrum centered around the carrier wave frequency k_0, see Fig. 1.3.

[1]Remark: In the area of ultrafast optics, people can produce pulses with only a few cycles of the

In the case of a spatially wide pulse, only a few k-components around a central wavenumber k_0 contribute in $A(k)$. Hence, the most relevant part of the dispersion relation $\omega(k)$ is the one in the vicinity around k_0. Then one can study the behavior of the pulse approximately by substituting the dispersion relation by its Taylor expansion around k_0:

$$\Delta k = k - k_0 \tag{1.125}$$

$$\begin{aligned}\omega(k) &= \underbrace{\omega(k_0)}_{=\omega_0} + \Delta k \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0} + \frac{1}{2}(\Delta k)^2 \left.\frac{d^2\omega}{dk^2}\right|_{k=k_0} + \ldots \\ &= \omega(k_0) + \Delta k\, \omega'(k_0) + \frac{1}{2}\Delta k^2 \omega''(k_0) + \ldots\end{aligned} \tag{1.126}$$

Hence, we obtain that the pulse maximum moves with the group velocity:

$$U(x,t) \simeq \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}(k_0 x - \omega_0 t)} \int\limits_{-\infty}^{\infty} \mathrm{d}\Delta k\, U(k_0 + \Delta k) e^{\mathrm{i}(\Delta k(x - t(\omega'(k_0) + \Delta K \omega''(k_0)/2 + \ldots)))} \tag{1.127}$$

$$= \frac{1}{\sqrt{2\pi}} A\,(x,t)\, e^{\mathrm{i}(k_0 x - \omega_0 t)} \tag{1.128}$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{A\,(x,t)}_{\text{slowly varying envelope modulation}} \cdot \underbrace{\phi\,(x,t)}_{\text{fast Oszillation}} \tag{1.129}$$

To define the different velocities we follow a fixed point on the wave: ϕ_0 is the Phase attrivuted to that fixed point.

$$\phi(x,t) = k_0 x(t) - \omega_0 t = \phi_0 \tag{1.130}$$

$$\text{Phase velocity} \qquad v_{\mathrm{ph}} = x(t) = \frac{\omega_0 t}{k_0} - \frac{\phi_0}{k_0} \qquad [= c \quad \text{in Vacuum}] \tag{1.131}$$

$$\text{Group velocity} \qquad v_{\mathrm{gr}} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}_{\phi_0} = (\frac{\mathrm{d}k}{\mathrm{d}\omega})^{-}1_{\omega_0} = (\frac{\mathrm{d}\omega}{\mathrm{d}k})_{\omega_0} \qquad [= c \quad \text{in Vacuum}] \tag{1.132}$$

$$\text{Energy velocity} \qquad v_{\mathrm{E}} = \frac{\langle S \rangle}{\langle w \rangle} \qquad [= c \quad \text{in Vacuum}] \tag{1.133}$$

carrier wave under the envelope. That is not a wide pulse, the envelope is only a few femto-seconds in time. Furthermore, in attosecond physics, people can now even produce "pulses" with single or even just half an optical cycle.

Very often $\omega(k) = \frac{ck}{n(k)}$, where $n(k)$ is the refractive index (backward wave tube):

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c}{n} - \underbrace{\frac{ck}{n^2(k)}}_{=\frac{\omega(k)}{n(k)}} \underbrace{\frac{\mathrm{d}n}{\mathrm{d}k}}_{\frac{\mathrm{d}n}{\mathrm{d}\omega}\frac{\mathrm{d}\omega}{\mathrm{d}k}}, \tag{1.134}$$

$$= \frac{c}{n} - \frac{\omega}{n(\omega)} \frac{\mathrm{d}\omega}{\mathrm{d}k} \frac{\mathrm{d}n}{\mathrm{d}\omega}. \tag{1.135}$$

$$\Rightarrow \quad v_{\mathrm{gr}} = \frac{c}{n(\omega) + \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}} \tag{1.136}$$

Going beyond the first order in the Taylor expansion above shows that non-linear dispersion, i.e., $\omega(k) \neq v_{\mathrm{ph}} k$ leads to pulse distortion.

In media, neither v_{ph} nor v_{gr} or v_{E} need to stay below the vacuum speed of light c.

A simple illustration uses an interferometric setup with an absorbing medium in one arm that preferentially "eats off" from the tail of the pulse. Then the pulse *appears* to propagate with superluminal velocity (see Fig. 1.4), though only the maximum peak motion is superluminal.

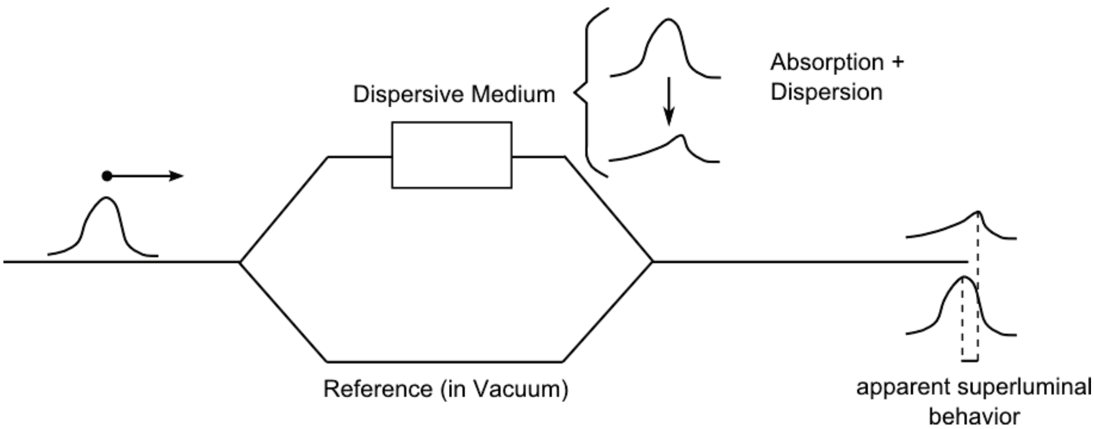


Figure 1.4: Scheme for observation of superluminal behavior.

Even negative phase and group velocities have been demonstrated experimentally. There is only *one* velocity that needs to obey $v < c$; this is the signal velocity (as already recognized by Sommerfeld!). In the above case, this would correspond to the velocity with which the rising front of the pulse propagates.

Nice visual examples for phase and group velocities can be found in the WWW on [2].