So,

$$\partial_n G = (\mathrm{i}k - \frac{1}{R})\frac{e^{\mathrm{i}kR}}{R} \tag{3.28}$$
$$= \mathrm{i}kG + \mathcal{O}(R^{-2}) \tag{3.29}$$
$$\underset{R\to\infty}{\backsimeq} \mathrm{i}kG. \tag{3.30}$$

Note that the physical meaning of $R \to \infty$ is really $R \gg |\mathbf{r}|$. Here, as discussed above, we got rid of all terms decaying faster than R^{-1}. Now, since U and hence, $\partial_n U$ is not known explicitly, we can only derive a condition that the outward normal derivative of U has to obey. Evaluating (3.24) only for the S_2 part now and using the expansion of the Green's function for large R we get

$$\iint_{S_2} \mathrm{d}s' \, [G\partial_n U - U \cdot (\mathrm{i}kG)] = \int_{\Omega_2} \mathrm{d}\Omega \underbrace{G}_{\approx\frac{1}{R} \text{ as} R\to\infty} (\partial_n U - \mathrm{i}kU)R^2 \tag{3.31}$$
$$\overset{R\to\infty}{\approx} \int_{\Omega_2} \mathrm{d}\Omega \; R(\partial_n U - \mathrm{i}kU). \tag{3.32}$$

Here we recast the surface integral in spherical coordinates as the integral over the solid angle Ω (basically, we performed a substitution of integration variables), where Ω_2 is the set of angles θ and φ defining the spherical shell S_2 with constant radius R. However, these details do not really matter, since this integral will vanish (become 0) for $R \to \infty$, if the integrand vanishes identically on the surface S_2, i.e. if

$$\boxed{\lim_{R\to\infty} R(\partial_n U - \mathrm{i}kU) = 0.} \tag{3.33}$$

This is the required boundary condition that U has to obey which is known as the *Sommerfeld radiation condition* or *outgoing wave condition*, because outward propagating spherical waves indeed fulfill this condition. This was alluded to above and finally puts all our assumptions about U on solid ground (as long as U fulfills that condition). Thus, with the Sommerfeld radiation condition as a boundary condition for U, all that remains in (3.24) is the contribution from the plane S_1 and for diffraction problems involving a screen, we then have

$$\boxed{U(\mathbf{r}) = \frac{1}{4\pi} \iint_{S_1} \mathrm{d}s' \, [(\partial_n U)G - U(\partial_n G)] + \text{ Sommerfeld radiation condition.}} \tag{3.34}$$

Kirchhoff considered an opaque screen with a pinhole and stated the following assumptions for this case: The major contributions to the integral $\iint_{S_1}$ stem from Σ (where the hole actually is), which is just a small portion of the plane S_1. Thus, on the screen, he introduced the *Kirchhoff boundary conditions*:

- Across the surface Σ the field distribution U and its derivative $\partial_n U$ are exactly the same as that would be in the absence of the screen (assuming illumination of the screen from the other side).

- Over the portion of S_1 that lies in the geometrical shadow of the screen, the field distribution U and its derivative $\partial_n U$ are identically zero (assuming illumination of the screen from the other side).

These conditions reduce (3.34) further to

$$U(\mathbf{r}) = \frac{1}{4\pi}\iint\limits_{\Sigma} \mathrm{d}s\, [(\partial_n U)G - U(\partial_n G)], \tag{3.35}$$

including Sommerfeld radiation condition,
using Kirchhoff boundary conditions.

Babinet's Principle:

If we replace the screen by a complementary screen where apertures and non-transparent parts are interchanged, the sum of the corresponding amplitudes U and U' are given by an integral over the total boundary plane S_1, resulting in

$$U(\mathbf{r}) + U'(\mathbf{r}) = U_0(\mathbf{r}), \tag{3.36}$$

where $U_0(\mathbf{r})$ is the undisturbed amplitude in the absence of the screens. This result is known as *Babinet's principle*.

For those points $\mathbf{r}$ where $U_0(\mathbf{r})$ vanishes, $U(\mathbf{r}) = -U'(\mathbf{r})$, i.e. the diffracted waves have opposite phases and equal absolute magnitudes, hence $|U(\mathbf{r})|^2 = |U'(\mathbf{r})|^2$.

Consequence

Suppose we have light that is sharply focused at a point $\mathbf{r}_0$ on the observer's surface F and the intensity $|U_0(\mathbf{r})|^2$ is essentially zero elsewhere for $\mathbf{r} \in F$. We can now use Babinet's principle in that region away from $\mathbf{r}_0$: If we put a diffracting object (e.g. a small disk/sphere) in the light path, the resulting diffraction pattern (i.e., the intensity $|U'(\mathbf{r})|^2$) will be the *very same* as for the complimentary object (i.e., a pinhole in a screen) everywhere on F except at the image point $\mathbf{r}_0$.

Applications:

- Measure size of objects (e.g., blood cells): Compare diffraction patterns of objects with those of holes of definite size (holes are easy to “drill”/etch!).
- Test validity of Kirchhoff's boundary conditions (See below)

Note:

There exists a rigorous derivation for Babinet's Principle from Maxwell's equations for the case where the complementary screens are made from ideal conductors (see Born and

Wolf, *Principles of Optics* [5]). Thus, Babinet's Principle has a wider range of validity than that of Kirchhoff's diffraction theory.

Now that the basic classical diffraction theory has been established by (3.35), we will study two approximations, that are widely used in practice for actual diffraction problems, namely the Fresnel and Fraunhofer approximations.

3.4 Fresnel-Kirchhoff Diffraction Formula

Simplification by Far-field Approximation:

Usually we have that $|\mathbf{r}-\mathbf{r}'| \gg \dfrac{1}{k}$, i.e. the observation point $\mathbf{r}$ is usually many optical wavelengths $\lambda = \frac{2\pi}{k}$ away from the aperture at $\mathbf{r}'$. In this case, we can apply a far-field approximation ($\cos(\hat{\mathbf{n}}, \mathbf{r}-\mathbf{r}')$ is the cosine of the angle between $\hat{\mathbf{n}}$ and $\mathbf{r}-\mathbf{r}'$):

$$\partial_n G = \cos(\hat{\mathbf{n}}, \mathbf{r}-\mathbf{r}') \left(\mathrm{i}k - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{3.37}$$

$$\overset{|\mathbf{r}-\mathbf{r}'| \gg \frac{1}{k}}{\simeq} \cos(\hat{\mathbf{n}}, \mathbf{r}-\mathbf{r}') \mathrm{i}k \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.38}$$

Plugging this approximation into (3.35) we get

$$U(\mathbf{r}) \overset{|\mathbf{r}-\mathbf{r}'| \gg \frac{1}{k}}{\simeq} \frac{1}{4\pi} \iint\limits_{\Sigma} \mathrm{d}s' \; \left[\partial_n U(\mathbf{r}') - \mathrm{i}kU(\mathbf{r}') \cos(\hat{\mathbf{n}}, \mathbf{r}-\mathbf{r}') \right] \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.39}$$

If the illuminating wave from the left is a spherical wave originating in $\mathbf{r}''$ of the form

$$U(\mathbf{r}') = A \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|} \tag{3.40}$$

with complex amplitude A, we obtain the celebrated *Fresnel-Kirchhoff diffraction formula* as

$$U(\mathbf{r}') = \frac{A}{\mathrm{i}\lambda} \iint\limits_{\Sigma} \mathrm{d}s' \; \frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{r}-\mathbf{r}'|+|\mathbf{r}'-\mathbf{r}''|)}}{|\mathbf{r}-\mathbf{r}'| \, |\mathbf{r}'-\mathbf{r}''|} \frac{1}{2} \left(\cos(\widehat{\mathbf{n}}, \mathbf{r}-\mathbf{r}') - \cos(\widehat{\mathbf{n}}, \mathbf{r}'-\mathbf{r}'') \right), \tag{3.41}$$

in far-field approximation,
for illuminating spherical wave in $\mathbf{r}'$,

where we used $k = \frac{2\pi}{\lambda}$ and shifted the remaining factor of $\frac{1}{2}$ to the cosine functions for reasons that become clear later on.

Interpretation:

There is a nice symmetry between the illumination point $\mathbf{r}''$ and the observation point $\mathbf{r}'$. A point source at $\mathbf{r}''$ will produce at $\mathbf{r}$ the same effect that a point source of equal intensity placed at $\mathbf{r}$ will produce at $\mathbf{r}''$.
Thus, we have the reciprocity theorem of Helmholtz

$$U(\mathbf{r}) = \iint_{\Sigma} \mathrm{d}s' \, \tilde{U}(\mathbf{r}') \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{3.42}$$

with

$$\tilde{U}(\mathbf{r}') := \frac{A}{\mathrm{i}\lambda} \frac{\mathrm{e}^{\mathrm{i}k(|\mathbf{r}'-\mathbf{r}''|)}}{|\mathbf{r}'-\mathbf{r}''|} \frac{1}{2} \left(\cos(\widehat{\mathbf{n}}, \mathbf{r}-\mathbf{r}') - \cos(\widehat{\mathbf{n}}, \mathbf{r}'-\mathbf{r}'')\right) \tag{3.43}$$

and can compare this with Fresnel's ad-hoc Ansatz in (3.2).

3.5 Rayleigh-Sommerfeld Formulation of Diffraction

Now we face a bit of a problem: The Kirchhoff formulation of diffraction is widely successful but it <u>cannot</u> be correct.
In particular, it is a well-known theorem of potential theory that if a two (or three) dimensional potential function *and* its normal derivative vanish *together* along a finite curve (surface) segment, then that potential function must vanish over the entire plane (volume).

Note:

The fact that one theory is consistent and the other is not, is a fundamental statement regarding the nature of the theories involved.
This does not necessarily mean that the former is more accurate than the latter.[2]

Let's summarize the basic diffraction theory we have used so far. We started with the theorem of Helmholtz and Kirchhoff in the form of

$$U(\mathbf{r}) = \frac{1}{4\pi} \iint_{S_1} \mathrm{d}s' \, (\partial_n U G - U \partial_n G) \tag{3.44}$$

where the following conditions had to be satisfied for the validity of this equation

- The scalar theory holds.
- U and G satisfy the homogeneous scalar wave equation (i.e., the Helmholtz equation).

[2] For example, the Newtonian theory of motion is not consistent with special relativity, yet quite successful in predicting the motion of cars and bicycles. We could cite thousands of other examples of inconsistencies in physical theories, which this work is too narrow to contain.

- The Sommerfeld radiation condition is satisfied.

Now suppose that the Green's function G of the Kirchhoff theory were modified such that,

- the derivation of the formula remains intact
- either G or $\partial_n G$ vanishes over the entire surface S_1.

Then, the necessity of imposing boundary conditions on *both* U and $\partial_n U$ would be removed, hence the inconsistencies of the Kirchhoff theory would be eliminated. Clearly, for a planar screen this is easily accomplished with an adaptation of the method of image charges from electrostatics to the present case of the Helmholtz equation. This has been realized by Sommerfeld (see Fig. 3.5):

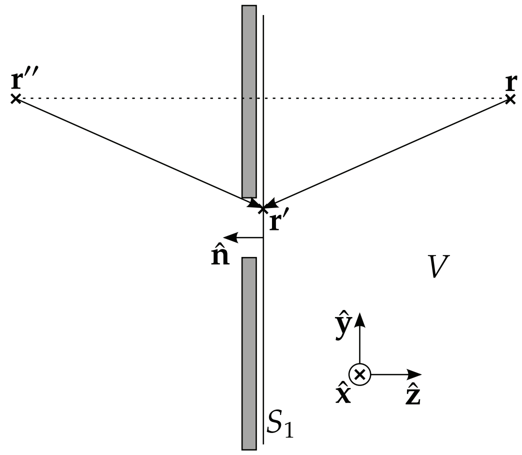

Figure 3.5: Point of observation $\mathbf{r}$ and its mirror location $\mathbf{r}''$. In the given coordinate system with the origin lying somewhere in the aperture plane S_1 (defining $z = 0$), for $\mathbf{r} = (x, y, z)$ the mirror point is $\mathbf{r}'' = (x, y, -z)$.

Given the original Green's function $G(\mathbf{r}, \mathbf{r}')$, we may add the function $G(\mathbf{r}'', \mathbf{r}')$ with the same phase (+) or with opposite phase (-) of the original Green's function, yielding

$$G_\pm(\mathbf{r}, \mathbf{r}') = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \pm \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}''-\mathbf{r}'|}}{|\mathbf{r}''-\mathbf{r}'|}. \tag{3.45}$$

As alluded to earlier, these new Green's functions obey the same differential equation (derivatives taken with respect to $\mathbf{r}' \in V$) in the volume V as $G(\mathbf{r}, \mathbf{r}')$. The behavior on the left side of the screen is not important for the diffraction problem in V. However, these Green's functions have the nice properties

$$G_-(\mathbf{r}, \mathbf{r}') = 0 \text{ for } \mathbf{r}' \in S_1, \qquad \partial_n G_+(\mathbf{r}, \mathbf{r}') = 0 \text{ for } \mathbf{r}' \in S_1, \tag{3.46}$$

where the derivative is still taken with respect to $\mathbf{r}'$. Using these Green's functions in (3.44), we obtain the *1st Rayleigh-Sommerfeld solution* as

$$U_I(\mathbf{r}) = -\frac{1}{4\pi} \iint_{S_1} \mathrm{d}s' \, U(\mathbf{r}') \partial_n G_-(\mathbf{r}, \mathbf{r}') \tag{3.47}$$

and the *2nd Rayleigh-Sommerfeld solution* as

$$U_{II}(\mathbf{r}) = \frac{1}{4\pi} \iint_{S_1} \mathrm{d}s' \, \partial_n U(\mathbf{r}') G_+(\mathbf{r}, \mathbf{r}'). \tag{3.48}$$

Both are now consistent with potential theory.

It turns out that,

- The Kirchhoff solution is the arithmetic average of the two Rayleigh-Sommerfeld solutions.
- Kirchhoff and the two Rayleigh-Sommerfeld solutions are essentially the same provided that the aperture diameter is much greater than the wavelength.
- For small angles, all solutions agree.
- For circular apertures on the axis: differences between theories only close to the aperture.
- In some sense the Kirchhoff theory is more general than the Rayleigh-Sommerfeld theory. The latter requires that the diffracting screens be planar, while the former does not. In practise, most apertures are planar, though.
- The first Rayleigh-Sommerfeld solution is the most simple to apply.

3.6 The Angular Spectrum Method

(Monochromatic)

Suppose that a wave is incident on a transverse xy-plane propagating with a non-zero k_z-component. Let's look at the envelope function U in the xy-plane at $z = 0$:

$$U(x, y, 0) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \; A(k_x, k_y; z = 0) \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} \tag{3.49}$$

It is unambiguously given by its spatial Fourier transform A as

$$A(k_x, k_y; z = 0) = \iint_{-\infty}^{\infty} \mathrm{d}x \mathrm{d}y \; U(x, y, 0) \mathrm{e}^{-\mathrm{i}(k_x x + k_y x)} \tag{3.50}$$

A is called the *angular spectrum* of U. Note that it is a hybrid quantity in the sense that the first two arguments are from the wavenumber space and the third is from position space. Hence, we can view the function $A(k_x, k_y, z)$ as propagating along the z-direction in the same sense as U propagated in the diffraction problems in previous sections: Given $A(k_x, k_y, z = 0)$ in one plane, we want to find $A(k_x, k_y, z)$ in all other planes, which in turn defines $U(x, y, z)$ as the Fourier transform

$$U(x, y, z) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \; A(k_x, k_y; z) \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} \tag{3.51}$$

and

$$A(k_x, k_y; z) = \iint_{-\infty}^{\infty} \mathrm{d}x \mathrm{d}y \; U(x, y, z) \mathrm{e}^{-\mathrm{i}(k_x x + k_y y)} \tag{3.52}$$

Hence, $A(k_x, k_y, z)$ readily contains all information we need to determine the interference pattern of the diffracted wave. We determine this function now when its values are given in one plane at $z = 0$.

U satisfies Helmholtz Equation

$$\triangle U + k^2 U = 0, \tag{3.53}$$

and by inserting the Fourier transform involving A defined above we can exploit the fact that derivatives are mapped into products, i.e. we obtain

$$\partial_z^2 A(k_x, k_y; z) + \left(k^2 - (k_x^2 + k_y^2)\right) A(k_x, k_y; z) = 0. \tag{3.54}$$

This differential equation can easily be solved by

$$A(k_x, k_y; z) = A(k_x, k_y; z = 0)\mathrm{e}^{\mathrm{i}\sqrt{k^2 - (k_x^2 + k_y^2)}z} \tag{3.55}$$

where we chose that sign (of the two possible solutions) such that the wave propagates or decays in positive z-direction. These are the two cases to consider:

$k_x^2 + k_y^2 < k^2$, propagating waves: carry energy away from the plane $z = 0$.

$k_x^2 + k_y^2 > k^2$, evanescent waves: energy stays in vicinity of the plane $z = 0$.

Hence, for the latter case, this solution becomes the real decaying exponential function (means non-propagating, i.e. standing wave)

$$A(k_x, k_y; z) = A(k_x, k_y; z = 0)\mathrm{e}^{-\mu z} \tag{3.56}$$

with $\mu = \sqrt{(k_x^2 + k_y^2) - k^2}$. The above expression does not carry energy away from the aperture; waves are cut-off far away from the aperture.

Now, for large distances z (such that $\mathrm{e}^{-\mu z}$ is practically 0), the free space propagation of light looks like this in terms of the angular spectrum:

$$U(x, y, z) = \iint_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y \; A(k_x, k_y; z = 0)\mathrm{e}^{\mathrm{i}\sqrt{k^2 - (k_x^2 + k_y^2)}z} \tag{3.57}$$

$$\times \underbrace{\Theta\left(k - \sqrt{k_x^2 + k_y^2}\right)\mathrm{e}^{\mathrm{i}(k_x x + k_y y)}}_{\text{linear dispersive filter with finite bandwidth}} \quad , \tag{3.58}$$

where Θ denotes the Heaviside step function and cuts off all evanescent parts that cannot be observed in the far-field.

Now suppose that an infinite opaque screen with a diffracting structure is introduced in the plane $z = 0$. We define the amplitude transmittance function t_A of this aperture,

$$t_\mathrm{A}(x, y) = \frac{U_\mathrm{t}(x, y; z = 0)}{U_\mathrm{i}(x, y; z = 0)} = \frac{\text{transmitted field amplitude}}{\text{incident field amplitude}} \tag{3.59}$$

Hence, when $t_{\mathrm{A}}(x,y)$ of a particular diffracting structure is given, the transmitted field directly behind it is given in terms of the incident field as

$$U_{\mathrm{t}}(x,y;z=0) = U_{\mathrm{i}}(x,y;z=0)t_{\mathrm{A}}(x,y). \tag{3.60}$$

In terms of the angular spectra of the respective waves, this takes on the form of a convolution

$$A_{\mathrm{t}}(k_x,k_y;z=0) = A_{\mathrm{i}}(k_x,k_y;z=0) \underbrace{\otimes}_{\text{convolution}} T(k_x,k_y), \tag{3.61}$$

where T is

$$T(k_x,k_y) = \iint\limits_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\; t_{\mathrm{A}}(x,y)\mathrm{e}^{-\mathrm{i}(k_x x + k_y y)}. \tag{3.62}$$

Note:

The angular spectrum approach and the first Rayleigh-Sommerfeld solution yield identical predictions of diffracted fields. G.C Shesmann J.Opt.Soc **57**, 546 (1967).

3.7 Fresnel and Fraunhofer Diffraction

We derive now typical and widely used approximations for the diffraction integrals.

General Diffraction problem:

We have the following ingredients:

Aperture (pinhole) in an opaque screen at $z=0$: Plane coordinates (ξ,η)

Observation at $\mathbf{r}$: Position $\mathbf{r}=(x,y,z)$

As alluded to earlier, the $1st$ Rayleigh-Sommerfeld solution is the simplest one, hence we start with that one:

$$U(\mathbf{r}) = \frac{1}{\mathrm{i}\lambda}\iint\limits_{\Sigma} \mathrm{d}s'\; U(\mathbf{r}')\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\underbrace{\cos\theta}_{=z/|\mathbf{r}-\mathbf{r}'|} \tag{3.63}$$

$$= \frac{z}{\mathrm{i}\lambda}\iint\limits_{\Sigma} \mathrm{d}\xi\mathrm{d}\eta\; U(\xi,\eta)\frac{e^{\mathrm{i}kr}}{r^2}, \tag{3.64}$$

where

$$\begin{aligned} r &= \sqrt{z^2+(x-\xi)^2+(y-\eta)^2} \\ &= z\sqrt{1+\left(\frac{x-\xi}{z}\right)^2+\left(\frac{x-\eta}{z}\right)^2}. \end{aligned} \tag{3.65}$$

These equations are (analytically) difficult to tackle, so that we seek certain reasonable approximations. In practice, this is going to be a far-field limit, since usually diffractive elements are used in far-field setups. Thus, we retain only the leading orders in z of the distance r in the above expressions, which describe the significant contributions to the far-fields. For the z-component, we then have $z \gg x - \xi$ and $z \gg y - \eta$, leading to

leading order in amplitude part r^2:

$$r \simeq z, \tag{3.66}$$

leading order in phase factor $\mathrm{e}^{\mathrm{i}kr}$:

$$r \simeq z \left(1 + \frac{1}{2} \left(\frac{x-\xi}{z} \right)^2 + \frac{1}{2} \left(\frac{y-\eta}{z} \right)^2 \right). \tag{3.67}$$

Note that we expanded the very same expression r up to two different orders in z according to the terms that they contribute to. In the amplitude, no significant information is gained when including terms $\sim z^{-2}$, but the integral would still look ugly and hard to solve because of the parts involving ξ and η. For the phase factor however, we need to retain terms up to order of z^{-2}, otherwise we'd throw away too much relevant physics, since the phase factor *must* depend on x and y to show any interference pattern at all! This explains the term *leading order*: sufficient order such that the desired physical effects are still visible in the equations but easier to treat/discuss.

3.7.1 Fresnel Diffraction

Plugging these expansions into the expression for the diffracted far-field U yields the *Fresnel diffraction integral*,

$$U(\mathbf{r}) = \frac{e^{\mathrm{i}kz}}{\mathrm{i}\lambda z} \iint\limits_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{d}\eta \; U(\xi, \eta) \mathrm{e}^{\mathrm{i}\frac{k}{2z}[(x-\xi)^2 + (y-\eta)^2]} \tag{3.68}$$

which is basically a convolution

$$U(\mathbf{r}) = \iint\limits_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{d}\eta \; U(\xi, \eta) h(x-\xi, y-\eta), \tag{3.69}$$

$$h(x,y) = \frac{\mathrm{e}^{\mathrm{i}kz}}{\mathrm{i}\lambda z} \mathrm{e}^{\frac{\mathrm{i}k}{2z}\left(x^2+y^2\right)}, \tag{3.70}$$

of the equivalent Fourier transform form:

$$U(\mathbf{r}) = \frac{e^{\mathrm{i}kz}}{\mathrm{i}\lambda z} \mathrm{e}^{\frac{\mathrm{i}k}{2z}\left(x^2+y^2\right)} \iint\limits_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{d}\eta \; \left(U(\xi, \eta) \mathrm{e}^{\frac{\mathrm{i}k}{2z}\left(\xi^2+\eta^2\right)} \right) \mathrm{e}^{\frac{-\mathrm{i}k}{z}(x\xi + y\eta)}. \tag{3.71}$$

General conclusion:

Despite its appearance, the accuracy of the Fresnel approximation is extremely good at distances that are very close to the aperture. W.H.Southwell J.Opt.Soc Am. **71**, 7 (1981)

3.7.2 Fraunhofer Diffraction

Fraunhofer diffraction assumes a stronger condition for the quadratic phase factor in the Fourier transform. This factor in the Fresnel integral is approximately unity over the aperture, when

$$z \gg \max_{\xi,\eta} \left(\frac{k}{2}(\xi^2 + \eta^2) \right). \tag{3.72}$$

Then the Fresnel integral simplifies to the *Fraunhofer diffraction integral*

$$U(\mathbf{r}) = \frac{e^{\mathrm{i}kz}}{\mathrm{i}\lambda z} \mathrm{e}^{\frac{\mathrm{i}k}{2z}\left(x^2+y^2\right)} \iint_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{d}\eta\, U(\xi,\eta) \mathrm{e}^{\frac{-\mathrm{i}k}{z}(x\xi+y\eta)}. \tag{3.73}$$

The conditions for validity of the Fraunhofer approximation can be severe. For example, for $\lambda = 0.6\,\mu\mathrm{m}$ and an aperture width of 2.5 cm, then $z \gg 1600\,\mathrm{m}$.

3.7.3 Examples of Fresnel Diffraction Patterns

We consider the transmission function

$$t_{\mathrm{A}}(\xi,\eta) \in \mathbb{C}. \tag{3.74}$$

It can

- modify amplitude: **Transmission modulation**,
- modify phase: **Phase modulation**.

Fresnel Diffraction by a Sinusoidal Amplitude Grating

A grating with period L and grating lines along the η-axis shall be described by

$$t_{\mathrm{A}}(\xi,\eta) = \frac{1}{2}\big(1 + m\cos(2\pi\frac{\xi}{L})\big). \tag{3.75}$$

The structure is assumed to be illuminated by a unit-amplitude, normally incident plane wave: Then the field $U(\xi,\eta)$ directly behind the grating is equal to $t_{\mathrm{A}}(\xi,\eta)$. By a completion of squares follows

$$U(\mathbf{r}) = \frac{1}{2}\big(1 + m\exp(-\mathrm{i}\pi\frac{\lambda z}{L^2})\cos(2\pi\frac{x}{L})\big), \tag{3.76}$$

$$I(\mathbf{r}) = |U(\mathbf{r})|^2 = \frac{1}{4}\big(1 + 2m\cos(\pi\frac{\lambda z}{L^2})\cos(2\pi\frac{x}{L}) + m^2\cos^2(2\pi\frac{x}{L})\big). \tag{3.77}$$

Analysis

case A: $\pi \frac{\lambda z}{L^2} = 2n\pi$

$$I(x, y) = \frac{1}{4}\big(1 + m\cos(2\pi\frac{x}{L})\big)^2. \qquad (3.78)$$

This is a perfect image of the grating. ⇒ Talbot Images (**Self-Images**, Henry Fox Talbot, 1836).

case B: $\pi \frac{\lambda z}{L^2} = (2n + 1)\pi$

$$I(x, y) = \frac{1}{4}\big(1 - m\cos(2\pi\frac{x}{L})\big)^2. \qquad (3.79)$$

These are phase-reversed Talbot Images (**Contrast Reversal**).

case C: $\pi \frac{\lambda z}{L^2} = (2n - 1)\frac{\pi}{2}$

$$I(x, y) = \frac{1}{4}\big((1 + \frac{m^2}{2}) + \frac{m^2}{2}\cos(4\pi\frac{x}{L})\big). \qquad (3.80)$$

This is a Talbot-Subimage (**half-period reduced contrast**).

These phenomena also occur in Bose-Einstein-Condensates (BECs), see Phys. Rev. A **51** (1), R14 (1996). For multichromatic versions, see Opt. Com. **260** (2), 415 (2005). For applications in distance measurements, see Meas. Sci. Technol. **111**, 77 (2000).

3.8 The Principle of Holography

We consider the image of an object that an observer sees through the pinhole in an otherwise opaque screen. If we could construct on the entire screen surface S a wave profile (amplitude and phase) that was an exact copy of the wave emitted by the object, the observer B could be placed arbitrarily and would not only see a mere picture of the object, but the object itself as if it were really there (and not just a screen showing the object). So instead of just recording the colors and intensities (only amplitudes, no phases) of the object with *photography*, we need to store and restore the amplitude and phase with *holography*.

If we could do this only on the pinhole Σ, the observer could “see” the object only for a certain range of angles. The practical construction of such a wave profile is extremely difficult.

⇒ Denis Gabor (for work done in the late 1940s) won the Nobel Prize in 1971 ”for his invention and development of the holographic method”.

The principle is depicted in Fig. 3.6 and Fig. **??**. The reference wave interferes on Σ with the object wave. The resulting interference pattern is made to expose a photosensitive material which changes its index according to the deposited dose/intensity. The object structure is now encoded into this so-called hologram.

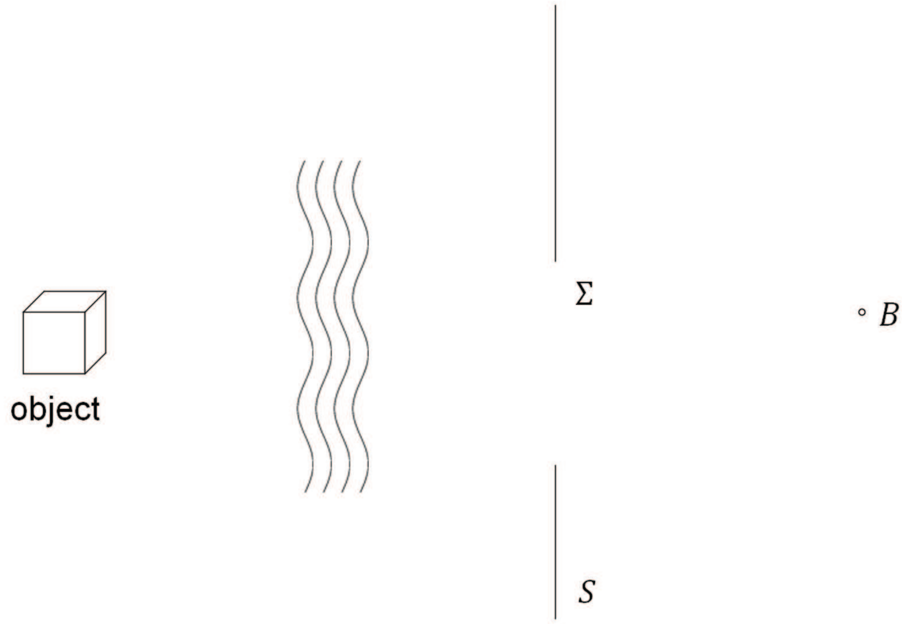

Figure 3.6: The object wave falling onto a screen Σ.

If the hologram is now exposed to the *same* reference wave, it acts as a diffractive object with a certain transmittance function. The combination of this reconstruction wave and transmittance function generates a wave, that recreates that from the original object in amplitude and phase so that an observer B sees a wave that is identical to that emitted from the original object. Hence, one cannot tell the difference and the object is perceived as if it was really there.