It has the following properties:

$$R(0) \geq 0, \tag{4.34}$$
$$R(-\tau) = R(\tau), \tag{4.35}$$
$$|R(\tau)| \leq R(0). \tag{4.36}$$

Note: If the mean $m(t)$ is not equal to zero, one often uses the centered auto-correlation function

$$\tilde{R}(t_1, t_2) = \langle (X(t_1) - m)(X(t_2) - m) \rangle \tag{4.37}$$

For complex processes Z(t), we obtain analogous definitions and properties with

$$R(\tau) = \langle Z^*(t) Z(t + \tau) \rangle, \tag{4.38}$$

and then we have

$$R(0) \geq 0, \tag{4.39}$$
$$R(-\tau) = R^*(\tau), \tag{4.40}$$
$$|R(\tau)| \leq R(0). \tag{4.41}$$

Cross-correlation Function:

For two random processes $Z_1(t), Z_2(t)$, we define the cross-correlation function as:

$$R_{12}(\tau) = \langle Z_1^*(t) Z_2(t + \tau) \rangle. \tag{4.42}$$

The cross-correlation function has the properties (stationarity assumed!):

$$|R_{12}(\tau)| \leq \sqrt{R_{11}(0) R_{22}(0)}, \tag{4.43}$$
$$R_{12}(-\tau) = R_{21}^*(\tau) \tag{4.44}$$

For instance, the cross-correlation function could be constructed for the field variables $U(\mathbf{r}, t)$ at two different points $\mathbf{r}_1$ and $\mathbf{r}_2$ in space.

Example: As an example for the auto-correlation function, we consider a finite sum of periodic components (sources) with random amplitudes. The associated random process is:

$$^k Z(t) = {}^k\xi_1 \mathrm{e}^{-\mathrm{i}\omega_1 t} + \ldots + {}^k\xi_m \mathrm{e}^{-\mathrm{i}\omega_m t} \tag{4.45}$$
$$= \sum_m^M {}^k\xi_m \mathrm{e}^{-\mathrm{i}\omega_m t}. \tag{4.46}$$

Here, each realization $^k Z(t)$ is labeled by k, where $^k\xi_m$ are M complex random variables that define the random amplitudes involved in the process. The amplitudes shall have the following properties

- $\langle {}^k\xi_m \rangle = 0 \, \forall m$: Each amplitude has zero mean.
- Each amplitude ${}^k\xi_m$ is itself a statistically stationary random process.

Then we have for the autocorrelation function

$$R(\tau) = \langle Z^*(t) Z(t+\tau) \rangle \tag{4.47}$$

$$= \left\langle \left(\sum_{m=1}^{M} \xi_m^* \mathrm{e}^{\mathrm{i}\omega_m t} \right) \left(\sum_{n=1}^{M} \xi_n \mathrm{e}^{-\mathrm{i}\omega_m (t+\tau)} \right) \right\rangle = \underbrace{\sum_{m,n}^{N} \langle \xi_m^* \xi_n \rangle \, \mathrm{e}^{-\mathrm{i}(\omega_n - \omega_m)t} \mathrm{e}^{-\mathrm{i}\omega_n \tau}}_{\text{must be independent of } t}. \tag{4.48}$$

Due to our assumption of stationarity, we known that $R(\tau)$ may no longer depend on t. Thus, we obtain the important result

$$\langle \xi_m^* \xi_m \rangle = 0 \text{ for} m \neq n, \tag{4.49}$$

i.e., the periodic terms of different frequencies must be *uncorrelated* and we have that

$$R(\tau) = \sum_{m=1}^{M} \langle \xi_m^* \xi_m \rangle \mathrm{e}^{-\mathrm{i}\omega_m \tau}. \tag{4.50}$$

In other words: $R(\tau)$ is a sum of periodic terms of frequencies $\omega_1, ..., \omega_M$ (as many as are present in the sample functions). The spectral component of each frequency ω_m is proportional to the average "energy" (average "power") associated with each periodic component in the process. *The auto-correlation function does not provide information about the phases of the periodic components.* Furthermore, strictly speaking, the above example process is not ergodic since for ergodicity to hold, we must have that $R(\tau) \to 0$ as $\tau \to \infty$, which is not given here.

This last deficit can be removed by considering the continuum limit, where the sum over the frequency components becomes an integral and we obtain Fourier integrals:

$$\langle Z(t) \rangle = 0 \tag{4.51}$$

$${}^k Z(t) = \int_{-\infty}^{\infty} d\omega \quad {}^k\xi(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \tag{4.52}$$

$${}^k\xi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \quad {}^k Z(t) \mathrm{e}^{\mathrm{i}\omega t} \tag{4.53}$$

$${}^k\xi^*(\omega) {}^k\xi(\omega') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \quad {}^k Z^*(t) {}^k Z(t') \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{e}^{\mathrm{i}\omega' t'} \tag{4.54}$$

$$t' = t + \tau \tag{4.55}$$

$$\langle \xi^*(\omega)\xi(\omega')\rangle = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau \underbrace{R(\tau)}_{=\langle Z(t)Z(t+\tau)\rangle} \mathrm{e}^{\mathrm{i}(\omega-\omega')t}\mathrm{e}^{\mathrm{i}\omega'\tau} \tag{4.56}$$

$$= S(\omega)\delta(\omega-\omega'). \tag{4.57}$$

Here, we used a standard representation for the Dirac delta function frequently encountered in physics,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{\mathrm{i}(\omega'-\omega)t} = \delta(\omega'-\omega). \tag{4.58}$$

The spectral function $S(\omega)$ associated with the correlation function $R(\tau)$ is

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau R(\tau)\mathrm{e}^{\mathrm{i}\omega\tau} \tag{4.59}$$

$$\Rightarrow R(\tau) = \int_{-\infty}^{\infty} d\omega S(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}. \tag{4.60}$$

Wiener-Khintchine Theorem:

The spectrum $S(\omega)$ of a stationary random process of zero mean and its auto-correlation function $R(\tau)$ form a Fourier-transform pair.

This can be generalized to cross-correlation functions

$$\langle \underbrace{\xi_1^*(\omega)}\xi_2(\omega')\rangle = W_{12}(\omega)\delta(\omega-\omega') \tag{4.61}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, {}^{k}Z_1(t)\mathrm{e}^{\mathrm{i}\omega t} \tag{4.62}$$

$$W_{12}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\tau\ R_{12}(\tau)\mathrm{e}^{\mathrm{i}\omega\tau} \tag{4.63}$$

$$R_{12}(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega\ W_{12}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}, \tag{4.64}$$

where $W_{12}(\omega)$ is the Cross-spectral density.

4.3 Stationary Optical Fields

Equipped with the above mathematical framework, we are now able to the quantify the degree of coherence. This is accomplished by introducing the *mutual coherence function* and the *complex degree of coherence*.

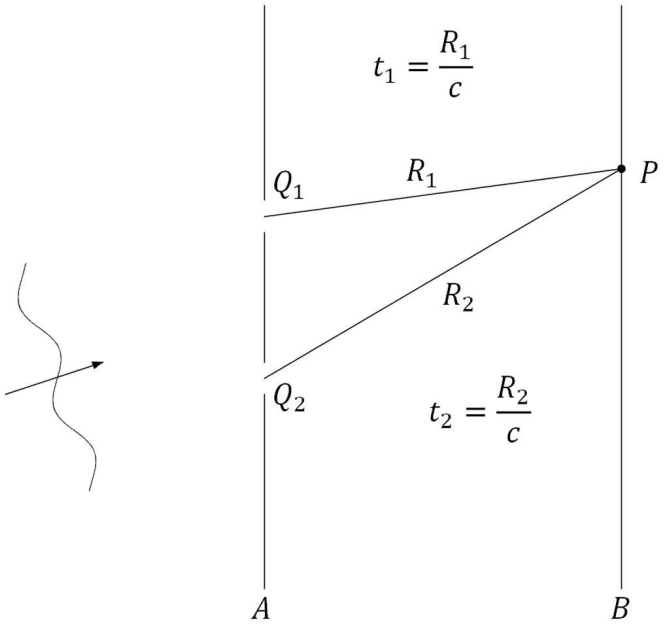

Figure 4.9: Schematic setup of Young's double slit experiment.

Young's Double-Slit Experiment:

The intensity $I(P)$ observed at a point P due to the illuminated pin holes at Q_1 and Q_2 is given as the average of the square of the sum of two fields (or, equivalently, two complex analytic signals) $V(Q_1, t - t_1)$ and $V(Q_2, t - t_2)$, where the times t_1 and t_2 are the result of different propagation lengths of the light paths R_1 and R_2 from Q_1 and Q_2 to P. In other words, we have

$$V(P,t) = K_1 V(Q_1, t - t_1) + K_2 V(Q_2, t - t_2) \tag{4.65}$$

where K_1 and K_2 take into account the (possible different) illumination strength and/or diffraction by the two different pin holes. As a result, we obtain for the observed intensity

$$\begin{aligned} I(P) &= \langle I(P,t) \rangle && (4.66)\\ &= \langle V^*(P,t) V(P,t) \rangle && (4.67)\\ &= |K_1|^2 \underbrace{\langle V^*(Q_1, t - t_1) V(Q_1, t - t_1) \rangle}_{= I(Q_1)} && (4.68)\\ &\quad + |K_2|^2 \underbrace{\langle V^*(Q_2, t - t_2) V(Q_2, t - t_2) \rangle}_{= I(Q_2)} && (4.69)\\ &\quad + 2\mathrm{Re}\big[K_1^* K_2 \underbrace{\langle V^*(Q_1, t - t_1) V(Q_2, t - t_2) \rangle}_{\text{Cross-correlation function} \quad \Gamma(Q_1, Q_2, \tau)} \big]. && (4.70) \end{aligned}$$

For a stationary process, we have

$$= \langle V^*(Q_1, t)V(Q_2, t+\tau)\rangle \tag{4.71}$$

Thus, we find for the intensity

$$I(p) = \underbrace{I^{(1)}(p)}_{=|K_1|^2 I(Q_1)} + \underbrace{I^{(2)}(p)}_{=|K_2|^2 I(Q_2)} + 2\mathrm{Re}\sqrt{I^{(1)}(p)}\sqrt{I^{(2)}(p)}\gamma(Q_1, Q_2, t_1 - t_2) \tag{4.72}$$

Here, The first two terms denote the intensity, if only pinhole 1 or 2 would be there.

We have introduced the complex second-order (in the fields) coherence function $\gamma(Q_1, Q_2, t_1 - t_2)$, according to

$$\gamma(Q_1, Q_2, t_1 - t_2) = \frac{\Gamma(Q_1, Q_2; \tau)}{\sqrt{I(Q_1)}\sqrt{I(Q_2)}} \tag{4.73}$$

$$= \frac{\Gamma(Q_1, Q_2; \tau)}{\sqrt{\Gamma(Q_1, Q_1, 0)\Gamma(Q_2, Q_2, 0)}}, \tag{4.74}$$

where $0 \leq |\gamma(Q_1, Q_2; \tau)| \leq 1$. Thus, γ serves as a measure of coherence:

$\gamma = 0$: absence of correlations (completely incoherent)

$\gamma = 1$: complete correlation (completely coherent)

In some cases, it is useful to separate the magnitude and phase of the second-order coherence function:

$$\gamma(Q_1, Q_2, \tau) = |\gamma(Q_1, Q_2, \tau)|\mathrm{e}^{\mathrm{i}\alpha(Q_1, Q_2, \tau) - \bar{\omega}\tau} \tag{4.75}$$

Where $\alpha = \bar{\omega}\tau - \arg\gamma(Q_1, Q_2, \tau)$.
Then, we have

$$I(p) = I^{(1)}(p) + I^{(2)}(p) + 2\sqrt{I^{(1)}(p)}\sqrt{I^{(2)}(p)}|\gamma(Q_1, Q_2, \tau)|\cos(\alpha - \delta) \tag{4.76}$$

We have $\delta = \bar{\omega}\tau = \bar{\omega}(t_2 - t_1) = \dfrac{2\pi}{\lambda}(R_2 - R_1)$ and $\lambda = \dfrac{2\pi c}{\bar{\omega}}$.

Furthermore, α varies slowly over τ-intervals that are short compared with the coherence time. Thus, if the change of $R_2 - R_1$ in the detection plane is small in comparison with the coherence length of the light, interference fringes (see Fig. 4.10) will be formed as a function of δ that have an amplitude $|\gamma|$. The associated fringe visibility $\gamma(p)$ is

$$\gamma(p) = \frac{I_{max}(p) - I_{min}(p)}{I_{max}(p) + I_{min}(p)} = |\gamma(Q_1, Q_2, \tau)| \tag{4.77}$$

It is also possible to consider the correlation functions for the intensity. The associated normalized interference term is known as the fourth-order coherence function (because two intensities, hence, four fields are involved).

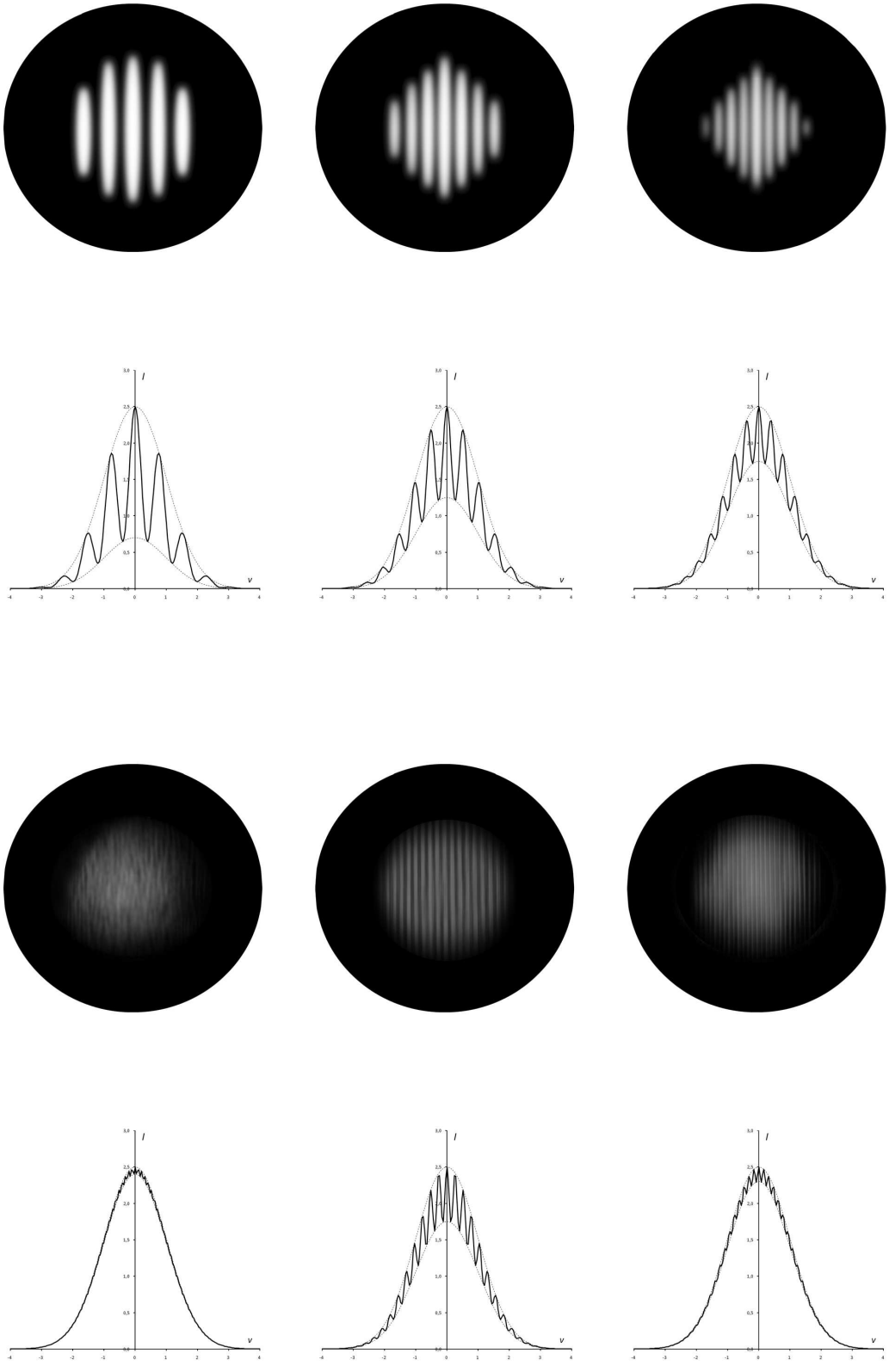

Figure 4.10: Patterns formed in Young's double-slit experiment, when using partially coherent quasi-monochromatic light.

The Van Cittert-Zernicke Theorem

We have already come across the different coherence length of the sun and Betelgeuse that we observe on Earth. The primary difference here is our vastly different distance from the two stars. As a matter of fact, one can prove that increasing the spatial separation from an incoherent source creates spatial coherence. This is the content of the van Cittert-Zernicke theorem which has a lot of implications and applications in projection systems and interferometric measurement techniques. We consider the spatial

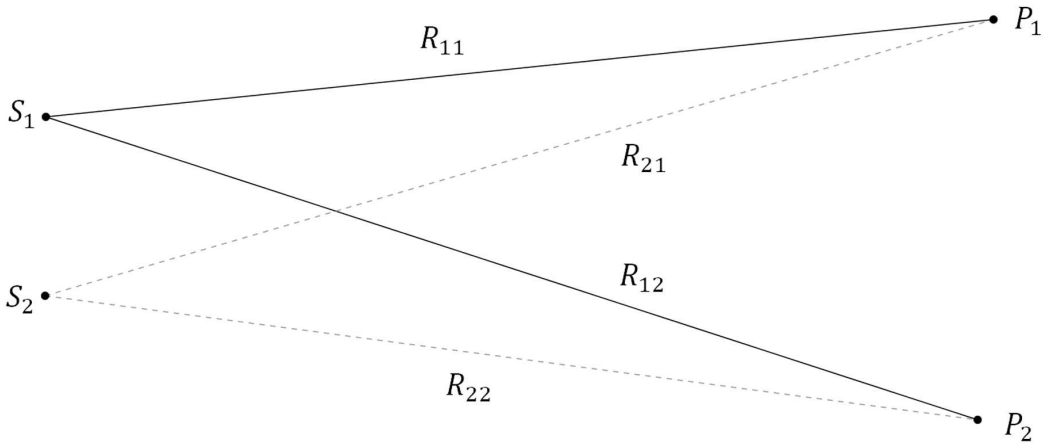

Figure 4.11: Schematic illustration of the situation considered in the van Cittert-Zernicke theorem.

coherence that is created at points P_1 and P_2 through the incoherent emission from two source points S_1 and S_2. If the following relations hold

$$|R_{11} - R_{12}| \ll l_c = \frac{2\pi c}{\Delta\omega} \quad \Rightarrow \quad V_1(P_2, t) \approx V_1(P_1, t) \tag{4.78}$$

$$|R_{22} - R_{21}| \ll l_c = \frac{2\pi c}{\Delta\omega} \quad \Rightarrow \quad V_2(P_2, t) \approx V_2(P_1, t) \tag{4.79}$$

we obtain that

$$V(P_1, t) = V_1(P_1, t) + V_2(P_1, t) \tag{4.80}$$

is correlated with

$$V(P_2, t) = V_1(P_2, t) + V_2(P_2, t) \tag{4.81}$$

although the sources S_1 and S_2 are uncorrelated!
In other words: For sufficiently large distances from a completely incoherent source, spatial coherence builds up.

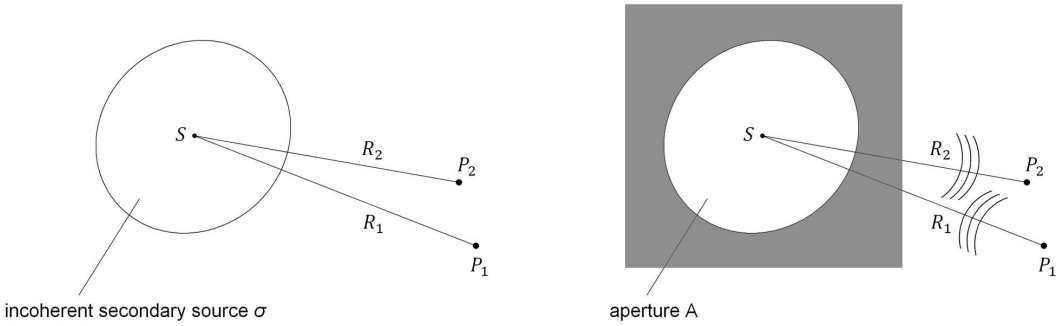

Figure 4.12: Illustration of the analogy between the van-Cittert-Zernicke theorem and the Huygens-Fresnel principle.

Detailed Analysis

In more detail, we consider the emission from a planar source and superimpose the different uncorrelated sources and obtain

$$\Gamma(P_1, P_2; 0) = \frac{1}{\sqrt{I(P_1)}\sqrt{I(P_2)}} \int_\sigma \mathrm{d}s \; I(s) \frac{\mathrm{e}^{\mathrm{i}\bar{k}(R_2 - R_1)}}{R_1 R_2} \tag{4.82}$$

In this and the subsequent integrals σ represents the domain of the source, i.e., the area where the emitters that contribute to the total measured signal are located.

The above integral (together with the definition of $I(s)$ below) is the celebrated van Cittert-Zernicke theorem. The above integral actually looks like a diffraction integral, and by now it should not come as a surprise that diffraction and coherence are related! Furthermore, we have introduced

$$I(P_i) = \int_\sigma \mathrm{d}s \; \frac{I(s)}{R_i^2} \tag{4.83}$$

The equation above represents the averaged intensity of P_1 and P_2.

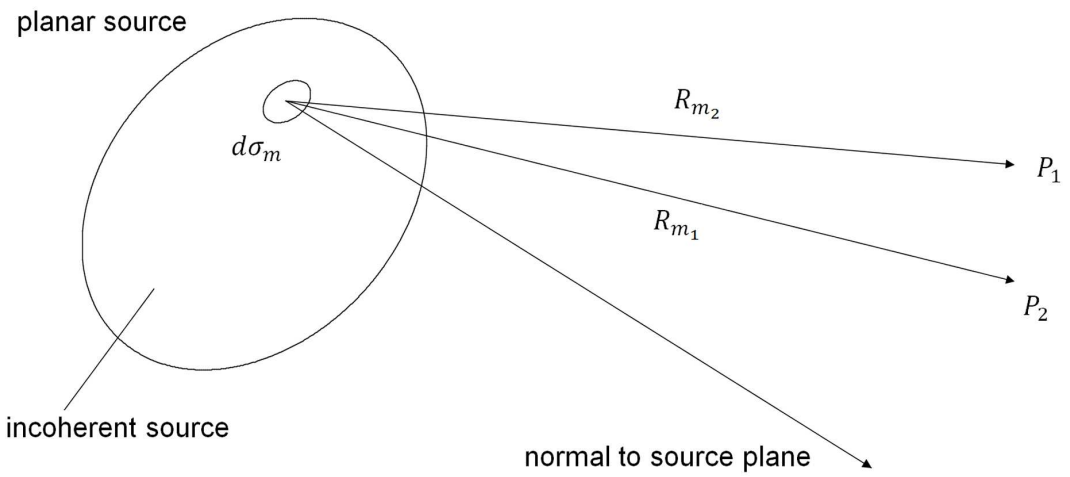

Figure 4.13: Radiation of an incoherent source as considered by the van Cittert-Zernicke theorem.

Applications:

Michelson Stellar Interferometer :

This instrument assumes a circular stellar disk with uniform emission: Then the fringe visibility vanishes for separations of the "pinholes" $d_0 = \frac{0.61\lambda_0}{\alpha}$, where α is the angular radius of the star. Therefore, we can obtain the size of a star from α and its distance to earth (see Fig. 4.14).

Optical (low coherence) Coherence Tomography :

The low coherence of a source in a Michelson interferometer provides an effective gating (see Fig. 4.15), which is necessary to obtain a good depth resolution. One observes interference only for arm length difference of the order of the effective coherence length. Depth-scanning is done via changing the reference arm length. This avoids the (rather formidable) ultrafast (time-) gating technology that would be required for a coherent (laser) source and replaces it with the low coherence time (length). However, there is no free lunch: Now the information is scrambled within the interference fringes and considerable image-processing is required. Nevertheless, in many applications an OCT setup is much more robust than a true (time-)gating setup and often there is sufficient time to do the image processing on a computer after the OCT image is recorded.

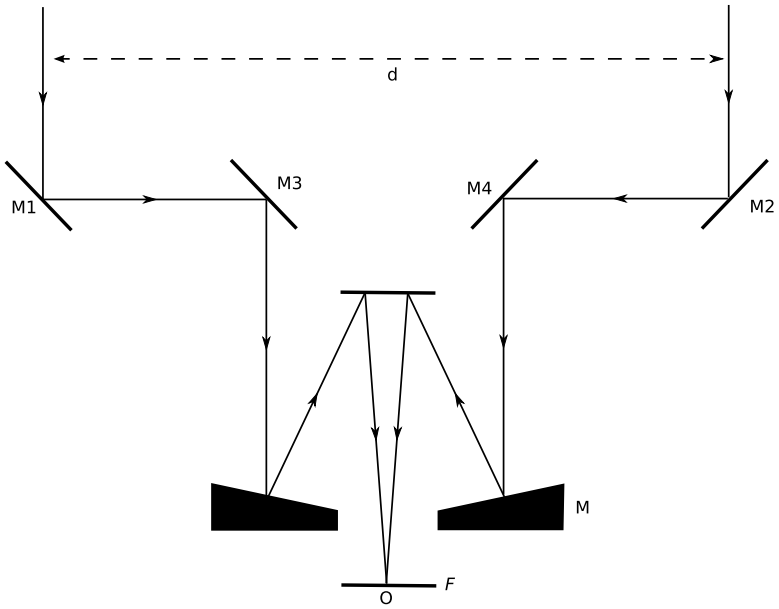

Figure 4.14: Sketch of a Michelson stellar interferometer.

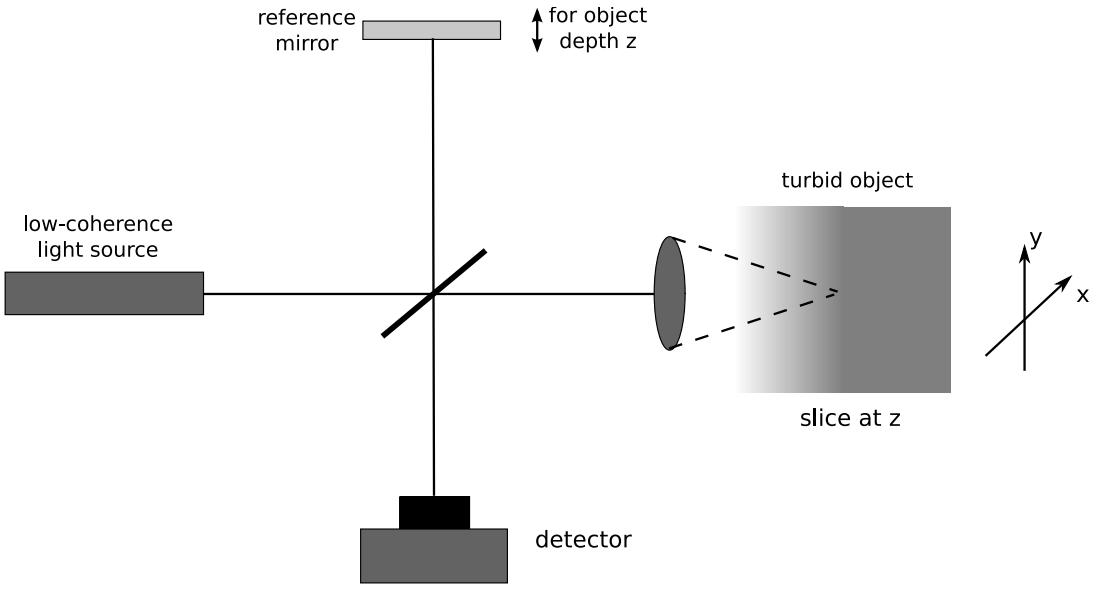

Figure 4.15: Sketch of the setup for imaging inside a turbid object using a Michelson design for optical coherence tomography.

5 Quantum Optics & Quantum Optical Coherence Theory

Until now, we have been concerned with the propagation of light according to the Maxwell equations. In order words, we have studied aspects of classical electromagnetic wave propagation. In this section, we develop the basics of the quantum theory of light with a special emphasis on the similarities and differences to what we have studied in the classical domain.

5.1 Review of Quantum Mechanics

First, we review the fundamental structure of quantum mechanics in the framework provided by the Schrödinger equation and its standard interpretation (aka Copenhagen interpretation; albeit an at least equally appropriate denomination would be the "Born interpretation", since Max Born has been awarded the 1954 Nobel Prize in Physics "for his fundamental research in quantum mechanics, especially for his statistical interpretation of the wavefunction"):

In quantum mechanics, a complex wave function, $\Psi(\mathbf{r}, t)$, describes the state of a system, say, a particle. This wave function does not have a direct physical interpretation.

Instead, $|\Psi(\mathbf{r}, t)|^2 \mathrm{d}^3 r$ represents the probability of finding the particle at time t in a volume of size $\mathrm{d}^3 r$ around $\mathbf{r}$. Obviously, a probability density has to be normalized so that we have

$$\int_{\mathbb{R}^3} \mathrm{d}^3 r |\Psi(\mathbf{r}, t)|^2 = 1. \tag{5.1}$$

The complex nature of the wave function allows for interference effects similar to those we have discussed in the earlier sections. However, since the interpretation of the quantum mechanical wave function is fundamentally different from that of a classical wave, significant differences occur (see below). In order to emphasize these points, many people refer to $\Psi(\mathbf{r}, t)$ as a *probability amplitude*.

In quantum mechanics, physical quantities (so-called *observables*) are given by Hermitian operators.[1] An operator $\hat{O}$ is said to be Hermitian if it coincides with its Hermitian

[1] An operator is just a mapping between functions, e.g. taking a derivative of a function, adding or

conjugate operator $\hat{O}^\dagger$. The Hermitian conjugated operator $\hat{O}^\dagger$ is defined through the following condition which $\hat{O}^\dagger$ has to satisfy for all complex and square-integrable functions $f(\mathbf{r})$ and $g(\mathbf{r})$:

$$\int \mathrm{d}^3 r \left(\hat{O}^\dagger f^*(\mathbf{r})\right) g(\mathbf{r}) = \int \mathrm{d}^3 r f^*(\mathbf{r}) \left(\hat{O}^\dagger g(\mathbf{r})\right) \qquad \forall f, g \tag{5.2}$$

Examples of observables are:
Energy $\rightarrow \mathrm{i}\hbar\partial_t$
Momentum $\rightarrow \frac{\hbar}{\mathrm{i}}\nabla$
Position $\rightarrow \mathbf{r}*$ (denoting the regular multiplication with $\mathbf{r}$)

Hermitian operators have

- real eigenvalues $\rightarrow$ this is what can be measured in a single measurement
- a complete and orthonormal set of eigenfunctions, i.e., the eigenvalue equation

$$\underbrace{\hat{O}}_{\text{operator}} \varphi(\mathbf{r}) = \underbrace{o}_{\text{number}} \varphi(\mathbf{r}) \qquad \text{(Eigenvalue equation)} \tag{5.3}$$

leads to a set of eigenvalues o_α and associated eigenfunctions $\varphi_\alpha(\mathbf{r})$.

With this, we can decompose any wave function $\Psi(\mathbf{r})$ according to the set of eigenfunctions of the observable $\hat{O}$ as

$$\Psi(\mathbf{r}) = \sum_\alpha c_\alpha \varphi_\alpha(\mathbf{r}) \tag{5.4}$$

and interpret $|c_\alpha|^2$ as the probability to measure the eigenvalue o_α given that the system is in state $\Psi(\mathbf{r})$. Note that the mathematics involved in this decomposition is analogous to an expansion of a regular vector into eigenvectors of a given Hermitian matrix.

The time evolution of a quantum mechanical system is given by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t \Psi(\mathbf{r}, t) = \hat{H}\Psi(\mathbf{r}, t). \tag{5.5}$$

Here, $\hat{H}$ represents the so-called Hamilton-operator which - in many cases - can be derived from the Hamilton function of the corresponding classical system. For instance, in classical mechanics, we often have a Hamilton function of the form

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{5.6}$$

where $V(\mathbf{r})$ is the potential energy. From this, we obtain the corresponding Hamilton-operator (aka *Hamiltonian*) via substitution of the (classical) momentum and potential through the corresponding (quantum mechanical) observables

$$\rightarrow \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{r}}) = -\frac{\hbar^2}{2m}\boldsymbol{\Delta} + V(\mathbf{r})* \tag{5.7}$$

multiplying a function with a constant. We are only concerned with linear operators, so any linear combination of the above mentioned operations yields a new operator.

The eigenvalues and -functions of the Hamiltonian $\hat{H}$ have a special significance since they correspond to the available eigenenergies (i.e., energy levels of atoms etc.) that are measurable. Furthermore, in many cases, the knowledge of the eigenvalues and -functions of the Hamiltonian facilitate the construction of solutions of the Schrödinger equation once the initial condition $\Psi(\mathbf{r}, t = t_0)$ at time t_0 is known.
Eigenvalues and -functions of the Hamiltonian satisfy

$$\hat{H}\varphi_n(\mathbf{r}) = E_n\varphi_n(\mathbf{r}), \tag{5.8}$$

so that we can (i) decompose the initial condition into the eigenfunctions and (ii) based on this, ca determine the full time-dependence of the wave function as

$$\varphi\left(\mathbf{r}, t_0\right) = \sum_n c_n\varphi_n(\mathbf{r}), \tag{5.9}$$

$$\Psi(\mathbf{r}, t) = \sum_n c_n \mathrm{e}^{-\mathrm{i}\frac{E_n t}{\hbar}}\varphi_n(\mathbf{r}). \tag{5.10}$$

In the above expression, we observe interference of several complex valued eigenfunctions (aka eigenstates) that changes with time.

The interpretation of the above can straightforwardly be generalized: $\Psi(\mathbf{r}, t)$ represents the *state of the system*, an abstract quantity that tells us which physical (actually observable) quantities can be measured in this state. If we wish to measure the observable $\hat{O}$, we determine the corresponding set of eigenvalues and -functions $\{o_\alpha\}, \{\varphi_\alpha(\mathbf{r})\}$. Then, we decompose the wave function into the eigenfunctions

$$\Psi(\mathbf{r}, t) = \sum_\alpha c_\alpha(t)\varphi_\alpha(\mathbf{r}) \tag{5.11}$$

so that

$$p_\alpha(t) := |c_\alpha(t)|^2 = |\int \mathrm{d}^3 r \varphi^*(\mathbf{r})\Psi(\mathbf{r}, t)|^2 \tag{5.12}$$

is the time-dependent (!) probability to measure the eigenvalue o_α.

To check this experimentally, we have to carry out a number of experiments on identical(!) systems that are all prepared in the same(!) state (i.e., are described by the same wave function Ψ). Then, the above probabilities translate into a histogram of relative occurrences of measured values, i.e., a distribution of measured values.

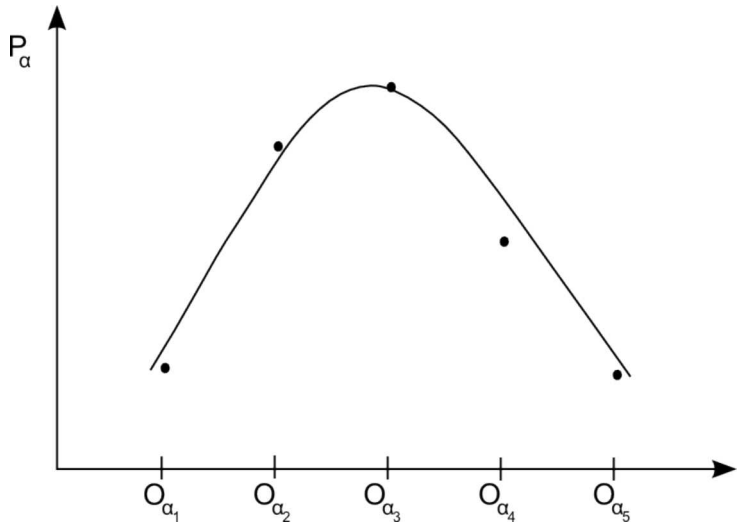

Figure 5.1: Distribution of measured values.

In classical physics we very often (but not always!) "see" only the average (or mean) value of this distribution. This so-called *expectation value* represents the classical value of the physical quantity (or observable) $\hat{O}$ in the state that is characterized by the wave function $\Psi(\mathbf{r}, t)$. In other words: In a given state, we assign a single value to each physical quantity according to

$$\langle \hat{O} \rangle(t) := \int \mathrm{d}^3 r \Psi^*(\mathbf{r}, t) \hat{O} \Psi(\mathbf{r}, t) \tag{5.13}$$

$$= \sum_\alpha \underbrace{|c_\alpha(t)|^2}_{=p_{\alpha(t)}} o_\alpha. \tag{5.14}$$

This is quite distinct, from the situation in classical physics (recall the corresponding discussion about classical physics in chapter 1).[2]

Owing to the probabilistic interpretation of quantum mechanics, we can also consider other things than expectation values. In this sense, quantum mechanics is infinitely richer than classical physics. For instance, we can consider the spread (or *variance*) of the distribution around the expectation value. In physics, this spread is called the uncertainty associated with observable $\hat{O}$ in state $\Psi(\mathbf{r}, t)$ and is defined as

$$\Delta O = \sqrt{\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2}. \tag{5.15}$$

If this value is large, the possible measurements can yield values that deviate (vary) strongly from the expectation value $\left\langle \hat{O} \right\rangle$. If it is small, measurements mostly yield

[2]The state/wave function determines the possible values and probabilities of physical quantities in a particular measurement. Figuratively speaking, a quantum measurement is like rolling a dice, where the state/wave function defines which dice to use: two-sided (aka a coin), six-sided, twelve-sided, twenty-sided, etc. Sounds weird but is true.

values in the vicinity of the expectation value. If the variance is zero, a measurement will definitely yield one single value (equal to the expectation value); the probabilities for all other values are plain zero then.

With this, we can derive the so-called generalized (Heisenberg) *uncertainty relation*

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \left[\hat{A}, \hat{B}\right] \rangle| \qquad \leftarrow \text{ depends on } \Psi! \tag{5.16}$$

$$\left[\hat{A}, \hat{B}\right] := \hat{A}\hat{B} - \hat{B}\hat{A} \qquad \text{(Commutator)} \tag{5.17}$$

The standard example for this uncertainty relation is the one for momentum and position (operators), where

$$[\hat{x}, \hat{p}_x] = x\frac{\hbar}{\mathrm{i}}\partial_x - \frac{\hbar}{\mathrm{i}}\partial_x x \tag{5.18}$$

$$= -\frac{\hbar}{\mathrm{i}} \tag{5.19}$$

$$= \mathrm{i}\hbar \tag{5.20}$$

leads to

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \text{independent of } \Psi \tag{5.21}$$

Up to here, the theory makes statements of a probabilistic nature which - as already alluded to above - have to be verified via repeated measurements on *identically* prepared systems.
This leads to somewhat counter-intuitive situations which can be found in many (more or less correct) popularizations, the prime example being "Schrödinger's Cat".

Finally, we have to define what happens after a measurement has been carried out, i.e., after we have obtained a definite (i.e., 100 percent certain) value. There is not much choice: After measuring the value o_α the system's wave function changes instantaneously (collapses) into the wave function of the corresponding eigenfunction. This is perhaps the most delicate part of the Copenhagen interpretation and a lot of criticism (violation of the principle of relativity due to the instantaneous nature of the collapse etc.) has been voiced plus alternative interpretations have been developed. However, to this day, all criticisms regarding violations of such principles have been cleared up and none of the alternative interpretations has even come near to describing all experiments let alone to making verifiable predictions that are different from the framework discussed above.

As an example with extremely high relevance to optics, we discuss the 1D harmonic oscillator:

$$\hat{H} := \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{5.22}$$

$$= -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2, \tag{5.23}$$

$$\hat{H}\varphi_n(\mathbf{r}) = E_n\varphi_n(\mathbf{r}), \tag{5.24}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N}_0, \tag{5.25}$$

$$\varphi_n(\mathbf{r}) = \sqrt{\frac{1}{2^n n!}}\, \mathrm{e}^{-\frac{m\omega}{2\hbar}x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \qquad \text{(Hermite functions)} \tag{5.26}$$

where H_n are the Hermite polynomials.

We define the important *ladder operators*:

$$\hat{a} = \frac{\hat{p}}{p_0} + \mathrm{i}\frac{\hat{x}}{x_0}, \qquad \text{with} \quad x_0 = \sqrt{\frac{\hbar}{m\omega}};\ p_0 = \sqrt{\hbar m\omega} \tag{5.27}$$

$$\hat{a}^\dagger = \frac{\hat{p}}{p_0} - \mathrm{i}\frac{\hat{x}}{x_0}. \tag{5.28}$$

These are non-Hermitian operators (and as such, no observables). The term ladder operators becomes clear by direct calculation

$$\hat{a}\varphi_n(\mathbf{r}) = \sqrt{n}\,\varphi_{n-1}(\mathbf{r}) \tag{5.29}$$

$$\hat{a}^\dagger\varphi_n(\mathbf{r}) = \sqrt{n+1}\,\varphi_{n+1}(\mathbf{r}) \tag{5.30}$$

$$\Rightarrow \varphi_n(\mathbf{r}) = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\varphi_0(\mathbf{r}) \tag{5.31}$$

The last equation means, that all eigenfunctions for the Hamiltonian of the harmonic oscillator can be obtained from one single function φ_0 and consecutive application of the raising (creation) operator $\hat{a}^\dagger$.

Since $\hat{a}$ is a non-Hermitian operator (note that obviously $\hat{a} \neq \hat{a}^\dagger$), it is not guaranteed that it possesses eigenvalues and -functions, let alone that any potential set of eigenfunctions would form a basis of the space of square-integrable functions. However, this is precisely the case and we find

$$\hat{a}\Psi_\alpha(\mathbf{r}) = \alpha\Psi_\alpha(\mathbf{r}) \tag{5.32}$$

$$\Rightarrow \Psi_\alpha(\mathbf{r}) = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\varphi_n(\mathbf{r}), \qquad \alpha \in \mathbb{C}. \tag{5.33}$$

For these states, we find

$$\int \mathrm{d}^3 r \Psi_\alpha^*(\mathbf{r})\Psi_\alpha(\mathbf{r}) = 1 \tag{5.34}$$

$$\int \mathrm{d}^3 r \Psi_\alpha^*(\mathbf{r})\Psi_\beta(\mathbf{r}) = \mathrm{e}^{-|\beta-\alpha|^2} \qquad \text{(overcomplete basis}^3\text{)} \tag{5.35}$$

Based on the above, we find the important commutation relation of the ladder operators

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar \Rightarrow \left[\hat{a}, \hat{a}^\dagger\right] = 1 \tag{5.36}$$

and can thus rewrite the Hamiltonian of the Harmonic Oscillator as

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \tag{5.37}$$

We also define the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$

$$\begin{aligned} \hat{n}\varphi_n(\mathbf{r}) &= \hat{a}^\dagger \hat{a} \varphi_n(\mathbf{r}) && (5.38)\\ &= \hat{a}^\dagger \sqrt{n} \varphi_{n-1}(\mathbf{r}) && (5.39)\\ &= n\varphi_n(\mathbf{r}), && (5.40) \end{aligned}$$

whose eigenvalues count the oscillation quanta in a given state. It can be used to write the Hamiltonian of the system in a different (better) form and see that the energy eigenvalues of the harmonic oscillator are uniformly spaced

$$\hat{H}\varphi_n(\mathbf{r}) = \hbar\omega \left(n + \frac{1}{2} \right) \varphi_n(\mathbf{r}) \tag{5.41}$$

with a lowest possible value of $E_0 = \hbar\omega\frac{1}{2} \neq 0$ (the zero-point energy).

In quantum optics (as in all higher applications of quantum mechanics) it has become customary to switch to a more abstract notation, the so-called *Dirac notation*, where - instead of the Hilbert space of square integrable functions - more abstract Hilbert spaces are considered. For instance, instead of the wave function $\Psi(\mathbf{r})$ (that characterizes a state) an abstract state vector $|\Psi\rangle$ is introduced such that

$$\underbrace{\Psi(\mathbf{r})}_{\in L^2(\mathbb{C})} := \langle \mathbf{r}|\Psi\rangle \qquad |\Psi\rangle \in \text{abstract Hilbert space} \tag{5.42}$$

We also define a scalar product $\langle\varphi|\Psi\rangle$ between two elements of the abstract Hilbert space according to

$$\begin{aligned} \int \mathrm{d}^3 r \varphi^*(\mathbf{r})\Psi(\mathbf{r}) &= \int \mathrm{d}^3 r \left(\langle \mathbf{r}|\varphi\rangle\right)^* \langle \mathbf{r}|\Psi\rangle && (5.43)\\ &= \int \mathrm{d}^3 r \langle\varphi|\mathbf{r}\rangle\langle\mathbf{r}|\Psi\rangle && (5.44)\\ &= \langle\varphi|\Psi\rangle \qquad \text{with} \quad \mathbb{1} = \int \mathrm{d}^3 r |\mathbf{r}\rangle\langle\mathbf{r}| && (5.45) \end{aligned}$$

Note that in this notation, $|\mathbf{r}\rangle\langle\mathbf{r}|$ represents a projection operator onto the abstract vector $|\mathbf{r}\rangle$ that acts on vectors of the abstract Hilbert space. $\mathbb{1}$ is the identity operator, the relation $\mathbb{1} = \int \mathrm{d}^3 r |\mathbf{r}\rangle\langle\mathbf{r}|$ is called *completeness relation*. It is a convenient way to express the identity in terms of the projection operators $|\mathbf{r}\rangle\langle\mathbf{r}|$.

Besides providing a highly economical notation, this reformulation allows us to reduce many complex problems that involve derivatives etc. to simple algebraic problems which eases quantum mechanical computations tremendously.

One more time, the prime example is the harmonic oscillator. We will review the results already discussed in terms of wave functions now in Dirac notation, where only algebraic relations matter. We repeat the essential steps below.

$$\varphi_n(x) = \langle x|n\rangle, \tag{5.46}$$
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{5.47}$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{5.48}$$
$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{5.49}$$
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{5.50}$$
$$\langle\alpha|\beta\rangle = \mathrm{e}^{-|\beta-\alpha|^2}, \tag{5.51}$$
$$\mathbb{1} = \int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha|. \tag{5.52}$$

Expectation values of operators $\hat{A}$ in a given state $|\Psi\rangle$ defined by the corresponding wave function $\Psi(\mathbf{r})$ are given by

$$\langle\hat{A}\rangle = \int \mathrm{d}^3r\,\Psi^*(\mathbf{r},t)\left(\hat{A}\Psi(\mathbf{r},t)\right) \tag{5.53}$$
$$= \langle\Psi|\hat{A}|\Psi\rangle. \tag{5.54}$$

Within the framework of the Dirac notation, the time evolution of a quantum mechanical state is described via the so-called *time evolution operator* $\hat{U}(t,t_0)$. It determines the state (and thus, the wave function) of the system at time t from a given state at time t_0:[4]

$$\mathrm{i}\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle \qquad \text{(Schrödinger equation)} \tag{5.55}$$
$$|\Psi(t)\rangle = \hat{U}(t,t_0)|\Psi(t_0)\rangle \qquad \text{(time evolution)} \tag{5.56}$$

We can distinguish several cases:

$\partial_t\hat{H} = 0$: Then $\hat{U} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\cdot(t-t_0)}$, i.e. the time evolution operator is the exponential function of the Hamiltonian multiplied by the passed time interval.

$\partial_t\hat{H} \neq 0$: Here we have to consider two cases:

[4] In classical mechanics, the state of an oscillator is given by definite values of position and momentum. A typical initial state is, e.g., an elongated spring held by a scientist ($x_0 \neq 0, p_0 = 0$). When let loose, these initial conditions cause the physical quantities x_0 and p_0 to assume time dependent values leading to the known oscillatory behavior determined by the classical equation of motion. In quantum mechanics, however, we deal with wave functions Ψ that define the probability to measure certain values of x and p. Here it is this probability distribution that changes with time and this change is determined by the quantum mechanical equation of motion, the Schrödinger equation, expressed in terms of the time evolution operator.

1. $\left[\hat{H}(t), \hat{H}(t')\right] = 0$: $\hat{U} = \exp(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\, \hat{H}(t'))$, i.e. the time evolution operator is the exponential function of the time integral of the time dependent Hamiltonian.
2. $\left[\hat{H}(t), \hat{H}(t')\right] \neq 0$: $\hat{U} = \hat{T}\cdot\exp(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t'\, \hat{H}(t'))$, with $\hat{T}$ being the time ordering operator.

In classical physics, the time-evolution is associated with the physical quantities. If we —loosely— associate the classical physical quantities with their quantum mechanical expectation values, we have

$$\langle \hat{A} \rangle(t) = \langle \hat{\Psi}(t) | \hat{A} | \hat{\Psi}(t) \rangle \tag{5.57}$$

However, we see that in the quantum mechanics we have discussed so far, the time evolution comes entirely from the time-dependence of the state vector and the physical quantities, i.e., the observables remain largely time-independent. This is the so-called *Schrödinger picture*: The time evolution is associated with the wave function (or state vector).
However, via the time evolution operator, we can equally well work with time-independent wave functions (or state vectors) and instead define time-dependent observables according to

$$\langle \hat{A} \rangle(t) = \langle \hat{U}(t, t_0)\Psi(t_0) | \hat{A} | \hat{U}(t, t_0)\Psi(t_0) \rangle \tag{5.58}$$

$$= \langle \Psi(t_0) | \underbrace{\hat{U}^\dagger(t, t_0)\hat{A}\hat{U}(t, t_0)}_{=\hat{A}(t)} | \Psi(t_0) \rangle. \tag{5.59}$$

In this so-called *Heisenberg picture*, the wave functions (state vectors) remain time-independent and the (physical) observables become time-dependent. More precisely, they obey the so-called *Heisenberg equation of motion*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}(t) = \frac{\mathrm{i}}{\hbar}\left[\hat{H}, \hat{A}(t)\right] + \underbrace{\partial_t \hat{A}(t)}_{=0 \text{ in most of our cases}}, \tag{5.60}$$

which —in the Heisenberg picture— replaces the Schrödinger equation for the state vectors. Of course, the Heisenberg and Schrödinger pictures are completely equivalent and it is only a matter of taste (or convenience) which one to use. In quantum optics, the use of the Heisenberg picture is standard.

5.2 Quantization of the EM-field

First, we consider the quantization of a single mode associated with an optical resonator of length L. Later on, we can generalize this to the case of multi-mode fields and consider the limit of $L \to \infty$.

Let the resonator be bounded by perfect mirrors that are located at $z = 0$ and $z = L$. Furthermore, let the electric field be polarized along the x-axis. By the boundary conditions, the classical modes in this "volume" $V = L$ take on the form

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q(t)\sin(kz), \tag{5.61}$$

$$k = k_m = \frac{\omega_m}{c} = m\frac{\pi}{L}, \quad m = 1, 2, \ldots \tag{5.62}$$

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\dot{q}(t)\cos(kz), \tag{5.63}$$

where $q(t)$ and $\dot{q}(t)$ are time dependent amplitudes, the strengths of the resonator modes. The classical Hamilton function for one of these modes can be determined by integrating the associated energy density over the "volume". Thus, we obtain

$$H = \frac{1}{2}\int \mathrm{d}V \left(\varepsilon_0\mathbf{E}^2(\mathbf{r},t) + \frac{1}{\mu_0}\mathbf{B}^2(\mathbf{r},t)\right) \tag{5.64}$$

$$= \frac{1}{2}(p^2 + \omega^2 q^2), \tag{5.65}$$

so q and $p = \dot{q}$ (unit mass) denote the canonical variables in the Hamilton formalism.

As described above, in many cases (and here, too) the quantization procedure substitutes the classical variables by Hermitian operators in a Hilbert space, that obey the position-momentum commutation relation:

$$q \mapsto \hat{q}, \tag{5.66}$$

$$p \mapsto \hat{p}, \tag{5.67}$$

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar, \tag{5.68}$$

With these substitutions, the Hamilton operator $\hat{H}$ for the electromagnetic field is given by a harmonic oscillator (which we conveniently discussed already):

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2). \tag{5.69}$$

As we have seen before, we may now introduce the so-called annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + \mathrm{i}\hat{p}) \tag{5.70}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - \mathrm{i}\hat{p}) \tag{5.71}$$

$$\tag{5.72}$$

which obey the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{5.73}$$

Then, the Hamiltonian takes the form of the quantum harmonic oscillator:

$$\hat{H} = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big). \tag{5.74}$$

The corresponding quantized EM-field operators are

$$\begin{aligned}
\hat{E}_x(z,t) &= E_0(\hat{a} + \hat{a}^\dagger)\sin(kz), && (5.75)\\
\hat{B}_y(z,t) &= B_0\frac{1}{\mathrm{i}}(\hat{a} - \hat{a}^\dagger)\cos(kz), && (5.76)\\
E_0 &= \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}, && (5.77)\\
B_0 &= \frac{\mu_0}{k}\sqrt{\frac{\varepsilon_0\hbar\omega^3}{V}}. && (5.78)
\end{aligned}$$

Several comments are in order:

1. The physical meaning of the creation and annihilation operator is that they, respectively, increase and decrease the number of energy quanta (the "photons") in the mode.

2. Since the mode profile (or "wave function" $\sin(kz)$) does not change in time, the time-dependence of the EM-field operators comes from the creation and annihilation operators. In other words: We naturally work in the Heisenberg picture where the creation and annihilation operators are time dependent: $\hat{a} \equiv \hat{a}(t)$ (and $\hat{a}^\dagger$ accordingly), as the following section shows.

Time Evolution of the Annihilation Operator $\hat{a}(t)$

In the Heisenberg picture, we employ the time evolution equation for the operators:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} &= \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{a}] && (5.79)\\
&= \frac{\mathrm{i}}{\hbar}\hbar\omega[(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \hat{a}] && (5.80)\\
&= \mathrm{i}\omega\Big(\hat{a}^\dagger\underbrace{[\hat{a}, \hat{a}]}_{=0} + \underbrace{[\hat{a}^\dagger, \hat{a}]}_{=-1}\hat{a}\Big) && (5.81)\\
&= -\mathrm{i}\omega\hat{a}. && (5.82)
\end{aligned}$$

Note that for actual computations such as the one shown above, it is useful to remember that operators commute with ordinary numbers and that the commutator exhibits a sort

of bilinear property (see the transition from (5.80) to (5.81) above).
The above simple ordinary differential equation has the solution

$$\hat{a}(t) = \hat{a}(0)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{5.83}$$

A similar analysis for the creation operator leads to

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)\mathrm{e}^{+\mathrm{i}\omega t}. \tag{5.84}$$

5.2.1 Quantum Fluctuation of a Single-Mode Field

Recall that the eigenstates of the Hamiltonian $\hat{H}$ are denoted by $|n\rangle$. The expectation values for the physically measurable EM-fields in these states can now be evaluated using only the algebraic relations of the harmonic oscillator:

$$\begin{aligned}
\langle n|\hat{E}_x(z,t)|n\rangle &= E_0 \sin(kz)\Big(\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle\Big) && (5.85)\\
&\equiv 0, && (5.86)\\
\langle n|\hat{E}_x^2(z,t)|n\rangle &= E_0^2 \sin^2(kz)\Big(\langle n|\hat{a}^{\dagger 2} + \hat{a}^2 + \hat{a}^\dagger\hat{a} + \underbrace{\hat{a}\hat{a}^\dagger}_{=\hat{a}^\dagger\hat{a}+1}|n\rangle\Big) && (5.87)\\
&= 2E_0^2 \sin^2(kz)(n+\frac{1}{2}). && (5.88)
\end{aligned}$$

Thus the uncertainty of the electric field is given as

$$\Delta\hat{E}_x(z,t) = \sqrt{2}E_0 \sin(kz)\sqrt{n+\frac{1}{2}}. \tag{5.89}$$

Recall, that the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ (which is a measure for the stored energy) does *not* commute with the electric field operator:

$$[\hat{n}, \hat{E}_x] = E_0 \sin(kz)(\hat{a}^\dagger - \hat{a}) \neq 0. \tag{5.90}$$

This results in an "Energy-Amplitude" uncertainty relation. In a particular state, this is given by

$$\Delta n \Delta E_x \geq \frac{1}{2}E_0 |\sin(kz)\left\langle \hat{a}^\dagger - \hat{a}\right\rangle| \tag{5.91}$$

This expression is analogous to the time-energy uncertainty.

Classical electromagnetic fields are determined by amplitude and phase. However, at this point, we want to note that the definition of a phase operator in quantum optics is not trivial.

Consequences of the above can best be illustrated when considering the output of a single-mode laser well above threshold. This emission is well described by a coherent state $|\alpha\rangle$ (i.e., the eigenstates of the annihilation operator), where

$$\begin{aligned}
|\alpha|^2 &\sim \text{averaged intensity} && (5.92)\\
\bar{n} &= \langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2. && (5.93)
\end{aligned}$$

Furthermore, we have

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \bar{n}^2 + \bar{n}. \tag{5.94}$$

The uncertainty associated with the number operator in this state is then given by

$$\Delta n = \sqrt{\bar{n}} \tag{5.95}$$

and we have

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \tag{5.96}$$

The probability P_n of detecting exactly n photons in such a coherent state is given by a Poisson distribution:

$$P_n = |\langle n|\alpha\rangle|^2 = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \tag{5.97}$$

In other words: A photon-counting experiment that determines how many photons are received within a certain time-interval will give a result with a probability according to Poisson-statistics. Many such experiments will lead to a statistics which - as the number of experiments is increased - approximate the Poisson statistics. A remarkable feature of the Poisson statistics is that for very large numbers (i.e., very large photon numbers which is the same as high intensity) the fluctuations of the photon numbers around the mean values are very small. This is equivalent to saying that a laser well above threshold has very little inherent "noise" (i.e., can in principle be made highly intensity-stable) and can very well be described by a classical field. In this sense, a coherent state with large values of $|\alpha|$ is "as classical as it gets". Other light sources cannot be described by a coherent state and, therefore, usually exhibit much more inherent "noise" (i.e., intensity fluctuations) and in some sense are less classical.

5.2.2 Extension to Multi-Mode Fields

The extension of the quantization procedure to multi-mode fields and higher dimension is now rather straightforward: We consider a cube of side-length L all whose faces are all perfect mirrors (i.e., a cubic resonator). Then, we obtain the allowed modes inside the cube as products of the modes discussed above for different directions, i.e., we obtain a wave vector instead of a single wave number. Since the modes are orthogonal, the integration of the energy density over the cube's volume will lead to a simple sum over

harmonic oscillators for each wave vector and polarization. Thus, we obtain

$$\hat{H} = \sum_{\mathbf{k}s} \hbar\omega_{\mathbf{k}s}(\hat{a}^\dagger_{\mathbf{k}s}\hat{a}_{\mathbf{k}s} + \frac{1}{2}), \tag{5.98}$$

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_{\mathbf{k}s} \mathbf{E}^{(0)}_{\mathbf{k}s}(\hat{a}_{\mathbf{k}s}(t) + \hat{a}^\dagger_{\mathbf{k}s}(t))\sin(\mathbf{k}\cdot\mathbf{r}), \tag{5.99}$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \sum_{\mathbf{k}s} \frac{1}{\mathrm{i}}\mathbf{B}^{(0)}_{\mathbf{k}s}(\hat{a}_{\mathbf{k}s}(t) - \hat{a}^\dagger_{\mathbf{k}s}(t))\cos(\mathbf{k}\cdot\mathbf{r}), \tag{5.100}$$

$$\mathbf{E}^{(0)}_{\mathbf{k}s} = \sqrt{\frac{\hbar\omega_{\mathbf{k}s}}{\varepsilon_0 V}}\hat{\mathbf{e}}_{\mathbf{k}s}, \tag{5.101}$$

$$\mathbf{B}^{(0)}_{\mathbf{k}s} = \frac{\mu_0}{|\mathbf{k}|}(\mathbf{k}\times\hat{\mathbf{e}}_{\mathbf{k}s})\sqrt{\frac{\varepsilon_0\hbar\omega^3_{\mathbf{k}s}}{V}}. \tag{5.102}$$

We introduce a short-hand notation for the bosonic Fock-states (multi-mode states) that are eigenstates of the above Hamiltonian:

$$|n\rangle \equiv |n_{\mathbf{k}_1 s_1}, n_{\mathbf{k}_1 s_2}, n_{\mathbf{k}_2 s_1}, n_{\mathbf{k}_2 s_2}, n_{\mathbf{k}_3 s_1}, \ldots\rangle \tag{5.103}$$

The creation/annihilation operators for a particular single-mode state $i \equiv (\mathbf{k}_i, s_i)$ act on these states as

$$\hat{a}_i\,|n_1,\ldots,n_i,\ldots\rangle = \sqrt{n_i}\,|n_1,\ldots,n_i-1,\ldots\rangle\,, \tag{5.104}$$

$$\hat{a}^\dagger_i\,|n_1,\ldots,n_i,\ldots\rangle = \sqrt{n_i+1}\,|n_1,\ldots,n_i+1,\ldots\rangle\,, \tag{5.105}$$

$$\hat{n}_i = \hat{a}^\dagger_i\hat{a}_i. \tag{5.106}$$

Here, $\hat{n}_i$ is the number operator for mode i. For the same mode i, the creation and annihilation operators obey the commutation relation as in the single mode case. For different modes i, j the creation and annihilation operators commute.

5.2.3 Density Operator, Pure and Mixed States

Very often, we know how much energy is (on average) contained in a light field but we do not know how this energy is distributed over the modes discussed above. In other words, many of the above modes can contribute and the best we can do is to assign a probability that that a particular mode contributes. In the end, we have to average over these probabilities.
This means that we have to introduce an ensemble (cf. ensemble vs. time-average) and other quantities that must be averaged over the probability p_i that the particular state $|\psi_i\rangle$ occurs in the ensemble. These probabilities obey

$$p_i \in \mathbb{R}, \tag{5.107}$$

$$\sum_i p_i = 1, \tag{5.108}$$

$$0 \leq p_i \leq 1. \tag{5.109}$$

Then the ensemble average of an operator's $\hat{A}$ expectation values is given by

$$
\begin{aligned}
[\hat{A}] &:= \sum_i p_i \langle\psi_i|\hat{A}|\psi_i\rangle && (5.110)\\
&= \sum_i p_i \langle\psi_i|\hat{A}\underbrace{\sum_n |\phi_n\rangle\langle\phi_n|}_{=\mathbb{1}}|\psi_i\rangle && (5.111)\\
&= \sum_n \langle\phi_n|\underbrace{\sum_i p_i |\psi_i\rangle\langle\psi_i|}_{=:\hat{\rho}}\hat{A}|\phi_n\rangle && (5.112)\\
&= \mathrm{Tr}(\hat{\rho}\hat{A}). && (5.113)
\end{aligned}
$$

Here, $\mathrm{Tr}(\cdot)$ represents the trace over the associated operator, i.e., a sum over its diagonal elements within any given basis.
Thus, we have introduced an operator $\hat{\rho}$ that describes the composition of the ensemble and this operator is called the *density operator*.

For a pure state, i.e., when the ensemble consists of only one single state denoted by j, we have

$$p_i = \delta_{ij} \tag{5.114}$$

so that for this case it follows

$$
\begin{aligned}
\hat{\rho}^2 &= \hat{\rho}, && (5.115)\\
\mathrm{Tr}\,\hat{\rho}^2 &= \mathrm{Tr}\,\hat{\rho} = 1. && (5.116)
\end{aligned}
$$

In contrast, for a mixed state (an ensemble comprising more than one particular state) we have

$$p_i \neq \delta_{ij}. \tag{5.117}$$

Then follows

$$
\begin{aligned}
\hat{\rho}^2 &= \sum_{ij} p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|, && (5.118)\\
\mathrm{Tr}\,\hat{\rho}^2 &= \sum_{ij} p_i p_j |\langle\psi_i|\psi_j\rangle|^2 && (5.119)\\
&\leq \Big[\sum_i p_i\Big]^2 = 1. && (5.120)
\end{aligned}
$$

Equality holds only if $|\langle\psi_i|\psi_j\rangle|^2 = 1$ for each pair of states ψ_i, ψ_j. This condition is only fulfilled, if

$$|\psi_i\rangle = \mathrm{e}^{\mathrm{i}\alpha}|\psi_j\rangle, \tag{5.121}$$

i. e., when all states are the same. The above is very practical criterion to figure out if the system described by a given density operator is a pure or mixed state:

$$
\begin{aligned}
\mathrm{Tr}\,\hat{\rho}^2 = 1 \qquad & \text{pure state}, && (5.122)\\
\mathrm{Tr}\,\hat{\rho}^2 < 1 \qquad & \text{mixed state}. && (5.123)
\end{aligned}
$$

5.2.4 Quantum Coherence Functions

The question is now how to connect or extend the classical theory of coherence to the quantum case. First, we recall the basics of classical coherence theory:

$$\gamma^{(1)}(x_1, x_2) = \frac{\langle E^*(x_1)E(x_2)\rangle}{\sqrt{\langle |E(x_1)|^2\rangle}\sqrt{\langle |E(x_2)|^2\rangle}}, \tag{5.124}$$

where

$$x_1 \equiv (\mathbf{r}_1, t_1), \quad x_2 \equiv (\mathbf{r}_2, t_2). \tag{5.125}$$

For stationary processes only time differences $t_1 - t_2$ matter.
The degree of coherence was classified as follows:

$$|\gamma^{(1)}| = 1 \qquad \text{complete coherence,} \tag{5.126}$$

$$0 < |\gamma^{(1)}| < 1 \qquad \text{partial coherence,} \tag{5.127}$$

$$|\gamma^{(1)}| = 0 \qquad \text{complete incoherence.} \tag{5.128}$$

Here, it is important to note that whenever the average over a product of two field values factorizes, we have complete coherence, i.e.,

$$\langle E^*(x_1)E(x_2)\rangle = \langle E^*(x_1)\rangle \langle E(x_2)\rangle \quad \Longrightarrow \quad \gamma^{(1)} \equiv 1. \tag{5.129}$$

The introduction of the notion of coherence in quantum optics is a non-trivial enterprise, since phase operators cannot easily be defined in quantum optics and the EM-field operators do not commute in general (which leads to quantum fluctuations which may or may not "interfere" with the statistical fluctuations that we have considered in classical coherence theory).
As a matter of fact, Roy J. Glauber was awarded the Nobel Prize in Physics 2005 "for his contribution to the quantum theory of optical coherence" in the 1960s. He suggested the (absolutely essential) decomposition of the EM-field operators according to

$$\hat{E}(\mathbf{r}, t) = \sum_{\mathbf{k}s} \mathbf{E}^{(0)}_{\mathbf{k}s}(\hat{a}_{\mathbf{k}s}(t)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \hat{a}^\dagger_{\mathbf{k}s}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}) \tag{5.130}$$

$$= \hat{E}^{(+)}(\mathbf{r}, t) + \hat{E}^{(-)}(\mathbf{r}, t), \tag{5.131}$$

$$\tag{5.132}$$

where

$$\hat{E}^{(-)} = \left[\hat{E}^{(+)}\right]^\dagger. \tag{5.133}$$

Then, one may define

$$G^{(1)}(x_1, x_2) = \mathrm{Tr}\left\{\hat{\rho}\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)\right\}, \tag{5.134}$$

where $x_1 = (\mathbf{r}_1, t_1)$ and $x_2 = (\mathbf{r}_2, t_2)$.
This has a direct connection to experiment: The intensity on a photodetector at position $\mathbf{r}$ is given by

$$I(\mathbf{r}, t) = G^{(1)}(x, x). \tag{5.135}$$

This is analogous to classical coherence theory, and we define the first order quantum coherence functions

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}}, \tag{5.136}$$

where again

$$|g^{(1)}(x_1, x_2)| = 1 \qquad \text{complete coherence,} \tag{5.137}$$
$$0 < |g^{(1)}(x_1, x_2)| < 1 \qquad \text{partial coherence,} \tag{5.138}$$
$$|g^{(1)}(x_1, x_2)| = 0 \qquad \text{complete incoherence.} \tag{5.139}$$

The function $G^{(1)}$ has the following properties:

$$G^{(1)}(x_1, x_2) = \left[G^{(1)}(x_2, x_1)\right]^*, \tag{5.140}$$
$$G^{(1)}(x, x) \geq 0, \tag{5.141}$$
$$G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2) \geq |G^{(1)}(x_1, x_2)|^2. \tag{5.142}$$
$$\tag{5.143}$$

Example: As an exercise, we consider a single-mode field:

$$|g^{(1)}(x_1, x_2)| = 1 \tag{5.144}$$

is obtained if the expectation value factorizes

$$G^{(1)}(x_1, x_2) = \left\langle \hat{E}^{(-)}(x_1) \hat{E}^{(+)}(x_2) \right\rangle \tag{5.145}$$
$$= \left\langle \hat{E}^{(-)}(x_1) \right\rangle \left\langle \hat{E}^{(+)}(x_2) \right\rangle. \tag{5.146}$$

This is the case for number states and coherent states. In other words for a single mode field, both number and coherent states exhibit perfect coherence. Other states generally do not.

However, the above quantity is still problematic, since at optical frequencies it is very hard to measure expectation (amplitude and phase!) values of the EM-field. It is much easier to measure intensities.

Recalling the Hanbury Brown and Twiss interferometer, we, therefore, consider the classical intensity correlation functions

$$\gamma^{(2)}(x_1, x_2) = \frac{\langle I(x_1) I(x_2) \rangle}{\langle I(x_1) \rangle \langle I(x_2) \rangle} \tag{5.147}$$
$$= \frac{\langle E^*(x_1) E^*(x_2) E(x_2) E(x_1) \rangle}{\langle |E(x_1)|^2 \rangle \langle |E(x_2)|^2 \rangle}. \tag{5.148}$$
$$\tag{5.149}$$

We have

$$1 \leq \gamma^{(2)}(0) < \infty, \tag{5.150}$$
$$\gamma^{(2)}(\tau) \leq \gamma^{(2)}(0). \tag{5.151}$$

5.2.5 Hanbury Brown and Twiss Expectation Values (Detection of Photons)

If light incident on one of the detectors is independent of light incident on the other, there should be a uniform coincidence rate, independent of t. Which value does this rate have? For light sources with large numbers of identically radiating atoms it can be shown:

$$\gamma^{(2)}(\tau) = 1 + |\gamma^{(1)}(\tau)|^2, \tag{5.152}$$
$$\Rightarrow \; 1 \leq |\gamma^{(2)}(\tau)| \leq 2, \tag{5.153}$$

since for incoherent light $\gamma^{(1)} \to 0$ as $\tau \to \infty$, the naive expectation has been that $\gamma^{(2)} \equiv 1$ throughout. However, Hanbury Brown and Twiss found experimentally for a thermal source

$$\gamma^{(2)}(0) = 2 \neq \gamma^{(2)}(\infty) = 1. \tag{5.154}$$

This means that a thermal source likely emits photons in pairs. This is known as the *photon bunching effect* and can not be explained in classical physics.

5.2.6 Generalization to Quantum Case

It is now straightforward to extend this to the quantum case. We introduce

$$G^{(2)}(x_1, x_2; x_2, x_1) = \mathrm{Tr}\left[\hat{\rho}\hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)\right], \tag{5.155}$$
$$g^{(2)}(x_1, x_2; x_2, x_1) = \frac{G^{(2)}(x_1, x_2; x_2, x_1)}{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)}. \tag{5.156}$$

If we consider fixed positions and investigate temporal coherence, we obtain

$$g^{(2)} \equiv g^{(2)}(\tau) = \frac{\left\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t) \right\rangle}{\left\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t) \right\rangle \left\langle \hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau) \right\rangle}. \tag{5.157}$$

This is nothing but the *conditional probability* that if a photon is detected at t, another one is also detected at $t + \tau$.

For a single mode field, we may evaluate this

$$
\begin{aligned}
g^{(2)}(\tau) &= \frac{\left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \right\rangle}{\left\langle \hat{a}^\dagger \hat{a} \right\rangle^2} && (5.158)\\
&= \frac{\left\langle \hat{n}(\hat{n}-1) \right\rangle}{\left\langle \hat{n} \right\rangle^2} && (5.159)\\
&= \frac{\left\langle \hat{n}^2 - \left\langle \hat{n} \right\rangle^2 \right\rangle + \left\langle \hat{n} \right\rangle^2 - \left\langle \hat{n} \right\rangle}{\left\langle \hat{n} \right\rangle^2} && (5.160)\\
&= 1 + \frac{(\Delta \hat{n})^2 - \left\langle \hat{n} \right\rangle}{\left\langle \hat{n} \right\rangle^2}. && (5.161)
\end{aligned}
$$

In a coherent state, we have $\Delta n = \sqrt{\bar{n}}$, thus

$$
g^{(2)}(\tau) \equiv 1 \text{ independent of } \tau. \tag{5.162}
$$

This result also holds for multimode fields and means, that photon arrival obeys Poisson statistics. However, thermal fields do not obey Poisson-statistics. Rather, one can show from statistical physics that for a single mode field:

$$
\begin{aligned}
\hat{\rho}_{\text{th}} &= \frac{\mathrm{e}^{-\hat{H}/k_B T}}{\operatorname{Tr} \mathrm{e}^{-\hat{H}/k_B T}} && (5.163)\\
&= \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n |n\rangle \langle n| , && (5.164)\\
\bar{n} &= \frac{1}{\mathrm{e}^{\hbar\omega/k_B T} - 1}, && (5.165)\\
\Delta n &= \sqrt{\bar{n} + \bar{n}^2}. && (5.166)
\end{aligned}
$$

and $g^{(2)}(0) = 2$ which verifies Hanbury Brown and Twiss.

For multi-mode fields one can actually obtain that $g^{(2)}(0)$ can be less than $g^{(2)}(\tau)$. This behavior cannot be understood classically and is known as *photon anti-bunching*.

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