Aufg 1. a

a) $D(\lambda) = e^{\lambda(\alpha a^\dagger - \alpha^* a)}$ Blatt 1

$$\frac{\partial}{\partial\lambda} D^\dagger(\lambda)\, a\, D(\lambda) = D^\dagger(\lambda)(-[\alpha a^\dagger - \alpha^* a])\, a\, D(\lambda)$$

$$+ D^\dagger(\lambda)\, a\, [\alpha a^\dagger - \alpha^* a]\, D(\lambda)$$

$$= -D^\dagger(\lambda)\,[\alpha a^\dagger - \alpha^* a, a]\, D(\lambda)$$

$$= \alpha \quad = \frac{\partial}{\partial\lambda} a(\lambda)$$

$$\int_0^1 \frac{\partial}{\partial\lambda} a(\lambda)\, d\lambda = \int_{a(0)} \int_0^1 \alpha\, d\lambda$$

$$a(1) - a(0) = \alpha$$

$$a(1) = a(0) + \alpha$$

$$D^\dagger a D = a + \alpha$$

b) $\alpha_1 = -\frac{f}{\hbar\omega}$

$$D^\dagger H D = \hbar\omega\left(a^\dagger - \frac{f}{\hbar\omega}\right)\left(a - \frac{f}{\hbar\omega}\right) + f\left(a^\dagger - \frac{f}{\hbar\omega} + a - \frac{f}{\hbar\omega}\right)$$

$$= \hbar\omega a^\dagger a$$

$$|n_e\rangle = D(\alpha)\,|n\rangle$$

b) Aufgabe b)

$D(\alpha) = D\left(-\frac{f}{\hbar\omega}\right)$

$|\psi'\rangle = D(\alpha)|\psi\rangle$

$D^{+} H D = \hbar\omega\left(a^{+}a - \frac{f}{\hbar\omega}\right)\left(a - \frac{f}{\hbar\omega}\right) + f(a + a^{+}) - \frac{f^2}{\hbar\omega}$

$= \hbar\omega a^{+}a$

$|n_D\rangle = D\left(-\frac{f}{\hbar\omega}\right)|n\rangle$

$H|n_D\rangle = E_n|n_D\rangle \qquad E_n = \hbar\omega n$

Aufg. 2: a) ~~[illegible]~~

$$S(\lambda) = \exp\left[\lambda\left(\xi^* a^2 - \xi a^{+2}\right)/2\right]$$

$$\frac{\partial}{\partial\lambda} S^+(\lambda)\, a\, S(\lambda) = -S^+\left[\left(\xi^* a^2 - \xi a^{+2}\right)/2 \,,\, a\right] S$$

$$S^+(\lambda)\, a\, S(\lambda) = a_\lambda$$

$$\frac{\partial}{\partial\lambda} a_\lambda = -\xi\, a^+_\lambda$$

$$\frac{\partial}{\partial\lambda} a^+_\lambda = -\xi^*\, a_\lambda \qquad \Rightarrow \qquad \frac{\partial}{\partial\lambda}\begin{pmatrix} a_\lambda \\ a^+_\lambda \end{pmatrix} = \begin{pmatrix} 0 & -\xi \\ -\xi^* & 0 \end{pmatrix}\begin{pmatrix} a_\lambda \\ a_\lambda \end{pmatrix}$$

Eigen w.: $\gamma_{1,2} = \pm |\xi| = \pm r \qquad \gamma_1 = r \quad \& \quad \gamma_2 = -r$

Eigen v.: $v_{1,2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{+i\theta} \\ 1 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} +e^{+i\theta} \\ 1 \end{pmatrix}$

Ansatz:

$$a_\lambda = \alpha_1 \left(-a e^{+i\theta} + a^+\right) e^{r\lambda} + \beta_1 \left(a^+ + e^{+i\theta} a\right) e^{-r\lambda}$$

$$a_{\lambda=0} = a \qquad \alpha_1 = -\beta \qquad \alpha_1 = -\frac{1}{2} e^{i\theta}$$

$$a_\lambda = \frac{1}{2} a \left(e^{r\lambda} + e^{-r\lambda}\right) - \frac{1}{2} a^+ e^{i\theta}\left(e^{r\lambda} - e^{-r\lambda}\right)$$

$$= a \cosh r - a^+ e^{i\theta} \sinh r$$

Test $\partial_\lambda a_\lambda = \alpha_1 r \left(a e^{-i\theta} + a^+\right) e^{r\lambda} - \beta_1 r \left(a^+ - e^{-i\theta} a\right) e^{-r\lambda}$

$$= -\xi a^+_\lambda = \xi \alpha_1^* \left(a^+ e^{i\theta} + a^*\right) e^{r\lambda} + \xi \beta_1 \left(a - e^{i\theta} a^+\right) e^{-r\lambda}$$

$$\xi \alpha_1^* e^{i\theta} = \alpha_1 r = \alpha_1^* e^{+2i\theta} r$$

$$\alpha_1^* e^{2i\theta} = \alpha_1 \quad \text{und} \quad \beta_1^* e^{2i\theta} = \beta_1$$

Aufg 2b (1)

$$S^\dagger(\xi)\, a^\dagger a\, S(\xi) = a^\dagger a\left(\cosh^2 r - \sinh^2 r\right) - \sinh^2 r$$
$$- \sinh r \cosh r \left[a^{\dagger 2} e^{i\theta} + a^2 e^{-i\theta}\right]$$

$$S^\dagger a^2 S = -\left[a^\dagger a + a a^\dagger\right] \cosh r \sinh r\, e^{-i\theta}$$
$$+ a^2 \cosh^2 r + a^{\dagger 2} \sinh r^2\, e^{-2i\theta}$$

$\theta = 0$

$$S^\dagger(\xi) H S(\xi) = \hbar\omega a^\dagger a - 4d\left[a^\dagger a + \tfrac{1}{2}\right]\cosh r \sinh r$$
$$- \hbar\omega \sinh r \cosh r \left[a^{\dagger 2} + a^2\right]$$
$$+ d\left(\cosh^2 r + \sinh^2 r\right)\left(a^2 + a^{\dagger 2}\right)$$

$$\hbar\omega \sinh r \cosh r = d\left(\cosh^2 r + \sinh^2 r\right)$$

$$\cancel{\hbar\omega(e^{-2r}\ \ e^{2r}}$$

$$\frac{1}{4}\hbar\omega\left(e^{2r} - e^{-2r}\right) = \frac{1}{2} d \left(e^{2r} + e^{-2r}\right)$$

$$e^{-4r} = \left(\frac{\hbar\omega - 2d}{\hbar\omega + 2d}\right) \qquad -4r = \ln\left(\frac{\hbar\omega - 2d}{\hbar\omega + 2d}\right)$$

$$\tilde{H} = S^\dagger H S = \left[\hbar\omega - d\left[\left(\frac{\hbar\omega + 2d}{\hbar\omega - 2d}\right)^{1/2} - \left(\frac{\hbar\omega - 2d}{\hbar\omega + 2d}\right)^{1/2}\right]\right] a^\dagger a$$

$$= \left[\hbar\omega - \frac{4 \cancel{\hbar\omega} d^2}{\sqrt{(\hbar\omega)^2 - 4d^2}}\right] a^\dagger a$$

$$\frac{1}{2} m \omega_d^2 q^2 = \frac{\hbar}{4} \frac{\omega_d^2}{\omega_0} \left(a^\dagger + a\right)^2 = \frac{\hbar\omega_d^2}{4\omega_0}\left(2 a^\dagger a + a^{\dagger 2} + a^2 + 1\right)$$
$$= 2d\, a^\dagger a + d\left(a^{\dagger 2} + a^2\right) + \dots$$

Aufg 2b (2)

$$\tilde{H} = \left[(\hbar\omega_0 + 2\alpha) - \frac{4\alpha}{\sqrt{(\hbar\omega_0+2\alpha)^2 - (4\alpha)^2}} \right] a^+ a$$

$$\approx \left[(\hbar\omega_0 + 2\alpha) - \frac{2\alpha^{3/2}}{\sqrt{\hbar\omega}} \right] a^+ a$$

$$2\alpha - \frac{2\alpha^{3/2}}{\sqrt{\hbar\omega_0}} = \frac{\hbar\omega\alpha^2}{2\omega_0} - \frac{\hbar\omega\alpha^3}{2\omega_0^2}$$

Aufg 3

$$H = \hbar\omega a^\dagger a$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega n t} |n\rangle$$

$$= D(\alpha e^{-i\omega t}) |0\rangle$$

$$= e^{\bar{\alpha}(t) \cancel{\alpha} a^\dagger - \bar{\alpha}^*(t) a} |0\rangle$$

$$= e^{\mathrm{Re}\,\bar{\alpha}(a^\dagger - a) + i\mathrm{Im}\,\bar{a}(a^\dagger + a)} |0\rangle$$

$$\sqrt{\frac{2m\omega}{\hbar}} q = (a^\dagger + a)$$

$$-i\sqrt{\frac{2}{\hbar m\omega}} p = (a^\dagger - a)$$

$$|\alpha(t)\rangle = e^{-i\hbar} \; e^{i\sqrt{\frac{2m\omega}{\hbar}} \mathrm{Im}(\bar{a}) q} \; e^{-i\sqrt{\frac{2}{\hbar m\omega}} \mathrm{Re}\,\bar{\alpha}\, p} |0\rangle$$

$$\frac{\partial}{\partial\alpha} e^{i\alpha p} q e^{-i\alpha p} = i e^{i\alpha p} [q, p] e^{i\alpha p}$$

$$= -\hbar \quad e^{i\alpha p} q e^{-i\alpha p} = q - \hbar\alpha$$

$$|\langle q | e^{-i\sqrt{\frac{2}{\hbar m\omega}} \mathrm{Re}\,\bar{\alpha}\, p} |0\rangle| \propto e^{-\frac{m\omega}{2\hbar}(q - q_0 \cos\omega t)^2}$$

$$\bar{\alpha} = \alpha e^{-i\omega t} \qquad q_0 = \sqrt{\frac{2\hbar}{m\omega}} \cancel{\alpha} \cancel{\mathrm{Re}}\, \alpha$$

reel

Acf g 4.

$$G(\lambda) \cancel{\mathbb{C}} = e^{\lambda A} e^{\lambda B}$$

$$\frac{\partial}{\partial\lambda} G(\lambda) = e^{\lambda A} [A + B] e^{\lambda B}$$

$$[e^{\lambda A}, B] = \sum_n \frac{1}{n!} \lambda^n [A^n, B]$$

$$[A^n, B] = n A^{n-1} [A, B]$$

$$n = 2 \quad [A^2, B] = A[A, B] + [A, B]A = 2A[A, B]$$

$$n \to n+1 \quad [A^{n+1}, B] = A[A^n, B] + [A, B]A^n$$

$$= n A A^{n-1} [A, B] + [A, B] A^n$$

$$= (n+1) A^n [A, B]$$

$$\Rightarrow [e^{\lambda A}, B] = \sum_n \frac{1}{n!} \lambda^n n A^{n-1} [A, B]$$

$$= \lambda [A, B] \sum_n \frac{1}{(n-1)!} \lambda^{n-1} A^{n-1}$$

$$= \lambda [A, B] e^{+\lambda A}$$

$$e^{\lambda A} B = [\lambda [A, B] + B] e^{+\lambda A}$$

$$\frac{\partial}{\partial\lambda} G(\lambda) = [A + B + \lambda [A, B]] e^{+\lambda A} e^{+\lambda B}$$

$$= [A + B + \lambda [A, B]] G(\lambda)$$

$$G(1) = e^{-(A + B + [A, B])/2} G(0)$$

$$e^{A} e^{B} = e^{+A + B + [A, B]/2}$$

$$e^{A} e^{B} e^{-[A, B]/2} = e^{+A + B}$$