Nonlinear wave interaction in photonic band gap materials

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Abstract

We present detailed analytical and numerical studies of nonlinear wave interaction processes in one-dimensional (1D) photonic band gap (PBG) materials with a Kerr nonlinearity. We demonstrate that some of these processes provide efficient mechanisms for dynamically controlling so-called gap-solitons. We derive analytical expressions that accurately determine the phase shifts experienced by nonlinear waves for a large class of non-resonant interaction processes. We also present comprehensive numerical studies of inelastic interactions, and show that rather distinct regimes of interaction exist. The predicted effects should be experimentally observable, and can be utilized for probing the existence and parameters of gap solitons. Our results are directly applicable to other nonlinear periodic structures such as Bose–Einstein condensates in optical lattices.

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1. Introduction

Electromagnetic waves propagating in periodically micro-structured dielectric materials, photonic crystals (PCs), share many properties of electron waves in ordinary semiconductors [1]. In particular, PCs exhibit multi-branch dispersion relations which may be separated by photonic band gaps (PBGs) [2]. In the frequency ranges of these gaps, linear waves decay exponentially with distance and ordinary (linear) wave propagation is prohibited. The existence of PBGs and the rich dispersive behavior near photonic band edges leads to numerous novel physical phenomena such as the inhibition of spontaneous emission of atoms [3], strong localization of light [4], photon-atom bound states [5], and super-refractive effects [6,7].

In the presence of a Kerr nonlinearity, the electromagnetic field intensity locally affects the refractive index of the constituent materials, which thus modifies the dispersion experienced by light. Consequently, for sufficiently intense fields nonlinear periodic structures may become transparent to electromagnetic waves with frequencies in the linear band gaps [8]. Starting with Winful et al. [9], a number of nonlinear effects, such as optical switching [10]...
and optical bistability [11], have been proposed theoretically, and observed experimentally.

One fascinating property of nonlinear PBG materials is the existence of so-called gap solitons, numerically discovered by Chen and Mills [12] in one-dimensional (1D) systems. The central frequency of a gap soliton lies within a PBG, and, perhaps more importantly, its propagation velocity can be arbitrarily small. The properties of gap solitons in one- and higher-dimensional nonlinear PBG materials have been the focus of much theoretical and experimental effort [13–18].

It has been shown [19–21] that the slowly varying envelope of an optical pulse with a carrier frequency within the vicinity of the PBG can be accurately described by the nonlinear Schrödinger equation (NLSE) [22]. To date, stationary gap solitons in optical systems have not been observed experimentally. However, Bragg solitons [23–25], which have a carrier frequency outside the PBG, and can travel at velocities much less than the speed of light in the background medium, have been observed in optical fiber Bragg gratings [26].

Although the NLSE is quite generally applicable if the nonlinear effects are sufficiently weak [21,27], it fails to be valid when the frequency of the light lies deep within the PBG. In such situations, light is described by the nonlinear coupled mode equations (NLCMEs) [28], which account for forward and backward propagating waves coupled by the Bragg scattering of the PBG material. The NLCMEs fully account for the linear dispersion of the system near the PBG. Thus, they permit the description of much shorter pulses than the NLSE model. Unfortunately, they are not integrable and, therefore, analytical tools for their analysis are strongly limited. However, for sufficiently wide pulses near the band edge, the NLCMEs can be reduced to the integrable NLSE model [21].

In this paper, we study theoretically the interaction of two pulses in Kerr nonlinear 1D PBG materials. We consider both non-resonant interaction [29], where the relative velocity between the two pulses is sufficiently large that the dynamics are insensitive to the relative phase of the interacting waves, and resonant interaction, where this phase becomes important. For non-resonant interaction, we provide details of the analytical calculations based on an NLSE approximation presented by some of us in a previous paper [29]. We then turn to extensive numerical simulations of pulse collisions using the NLCMEs, and examine the parameter regimes that lead to either resonant or non-resonant interactions.

A detailed understanding of the interaction of nonlinear waves provides numerous pathways to the dynamic control and manipulation of light by light, and may facilitate applications in optical buffers and delay lines. For instance, in the NLSE limit, the non-resonant collision of a Bragg with a gap soliton results in a phase-shift of the carrier waves which translates into a corresponding wavefront shift [29]. Thus, it becomes possible to employ Bragg solitons to control the position of stationary gap solitons; alternatively, Bragg solitons may be used as a probe to confirm the existence and determine the parameters of a stationary gap soliton [29], as schematically displayed in Fig. 1. For material parameters consistent with typical Bragg soliton experiments, the wavefront shift translates into time delays or advances of tens of picoseconds, which should be easily observable in the laboratory. Furthermore, nonlinear wave interaction processes might be useful for experimentally launching stationary gap solitons through successive collisions with Bragg solitons that would pull them into a PBG material [29]. We note that it was recently suggested to create stationary gap solitons through the collision of two mobile gap solitons [30].

This paper is organized as follows: in Section 2, we briefly describe 1D PBGs, and we give the NLSE description for a single pulse propagating in the presence of a Kerr nonlinearity. We also introduce the NLCSEs. Section 3 features a detailed investigation of the non-resonant interaction of nonlinear waves in PBG materials. In particular, we derive an analytical expression for the phase shift of a nonlinear optical pulse upon collision with another pulse. In Section 4, we present numerical results of the NLCMEs for resonant and non-resonant interactions and compare them with our analytical results. We identify several rather distinct interaction regimes. In Section 5 we conclude.

2. One-dimensional photonic band gap materials

The propagation of electromagnetic waves in photonic crystals is governed by Maxwell’s equations [31,32]. In this paper we consider 1D systems for which, under assumptions discussed below, Maxwell’s equations lead to the wave equation for the electric field, \( E(x,t) \):

\[
\left[ \frac{\partial^2}{\partial x^2} - \frac{\varepsilon(x)}{c^2} \frac{\partial^2}{\partial t^2} \right] E(x,t) = \frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} P_{NL}(x,t). \tag{1}
\]

In deriving Eq. (1) we assume the absence of free charges and currents. We also assume that the PC exhibits no magnetic response, because for PC materials of interest to us the relaxation time of a magnetic moment is several orders of magnitude larger than the time period of the optical waves [33]. However,
we include a linear polarization via the dielectric constant \( \varepsilon(x) \equiv n^2(x) \), where \( n(x) \) is the refractive index of the material; and a Kerr nonlinear polarization via:

\[
P_{NL}(x, t) = \chi^{(3)}(x)E^3(x, t),
\]

where \( \chi^{(3)}(x) \) is the Kerr nonlinear coefficient.

For a PC, both \( \varepsilon(x) \) and \( \chi^{(3)}(x) \) are real, periodic functions of \( x \), with a period \( a \). In principle, \( \varepsilon(x) \) is also frequency-dependent. However, we neglect the intrinsic material dispersion relative to the strong dispersion induced by the PC structure near a PBG. Since the dispersive and dissipative properties of dielectrics are related via the Kramers–Kronig relations [33], this implies that the PC is composed of low-loss dielectrics so that losses can (and will) be neglected.

In the absence of nonlinearity, the dispersion relation of a 1D PBG structure exhibits forbidden frequency regions, centred around so-called Bragg frequencies, in which light does not propagate. Frequencies, \( \omega_m \), that are not forbidden can be associated with Bloch functions, \( \varphi_m(x) \), where the composite index \( m \equiv (nk) \) denotes the band-index \( n \), and the wavenumber \( k \) of the dispersion relation. The Bloch functions form a complete orthonormal set with:

\[
\langle m | \varepsilon(x) | m' \rangle \equiv \int_{-\infty}^{\infty} \varphi^{*}_{m}(x)\varepsilon(x)\varphi_{m'}(x_0) \, dx_0 = \delta_{mm'},
\]

where the integration domain is the entire photonic crystal. Note that we have introduced a bra- and ket-notation.

If we introduce a weak nonlinearity, the wave equation (1) can be analyzed via a perturbation theory. Our approach is to apply the method of multiple scales to consider two pulses with slowly varying envelopes propagating simultaneously in the system. However, before treating this case directly, we briefly summarize some useful results for the case where only one pulse is propagating. We will return to the two-pulse case in the following section.

The case where one pulse with a slowly varying envelope propagates in a Kerr nonlinear PC has been thoroughly analyzed using the method of multiple scales. There are two main regimes of interest. First, when the carrier frequency of the pulse is either outside, or just slightly within, the band gap of the PC, then the pulse dynamics are well described by a NLSE of the form

\[
i \frac{\partial A}{\partial t} + \frac{1}{2} \omega_m \frac{\partial^2 A}{\partial x^2} + \chi^{(3)}_{\text{eff}} |A|^2 A = 0,
\]

where \( A \) is the envelope function of the electric field, carried at frequency \( \omega_m \), and \( \omega_m'' \equiv \partial^2 \omega_m / \partial k^2 \) is the group velocity dispersion (GVD) of the periodic medium at \( \omega_m \). Note that we have presented the NLSE in a Galilean frame of reference, travelling with the

Fig. 1. Illustration of the non-resonant interaction of a Bragg and a stationary gap soliton. This process causes a wavefront shift of both pulses which may be used to control the position, or to probe the existence and parameters, of the gap soliton.
group velocity \((\omega''_m)\) of the pulse. The effective nonlinear coefficient in (4) is:

\[ \chi_{\text{eff}}^{(3)} = 6\pi \omega_m \int \chi^{(3)}(x) |\varphi_m(x)|^4 \, dx. \]  

(5)

This overlap integral reflects the fact that the Bloch functions are not uniformly distributed over the constituent materials of a PC.

The NLSE (4) exhibits soliton solutions with a familiar sech amplitude profile when the Lighthill condition, \(\chi_{\text{eff}}^{(3)} \omega''_m > 0\), holds [34]. In Fig. 2 we show a typical dispersion relation for a 1D PC in the vicinity of a band gap. For frequencies above (below) the gap, \(\omega''_m\) is positive (negative). Therefore, for focusing (defocusing) nonlinearities, where \(\chi_{\text{eff}}^{(3)} > 0\) \((\chi_{\text{eff}}^{(3)} < 0)\), solitons can be formed only if the pulse carrier frequency is near the upper (lower) band edge.

The NLSE can describe pulses with a carrier frequency within the band gap. In this case the soliton solution is called a ‘gap soliton’. If the carrier frequency of the pulse is outside the gap, the soliton is referred to as a ‘Bragg soliton’. These two solution regimes are indicated in Fig. 2. It is well known that gap solitons can exhibit a vanishing propagation velocity \([19,28]\). Clearly, when the group velocity, \(v_g\), vanishes, no energy is transported and the carrier wave of the pulse is a standing wave, i.e., a Bloch function at the band edge. Nonlinear pulses with zero group velocity correspond to a standing wave, i.e., a Bloch function at the band edge.

\[ A(x, t) = A_1(x) \exp(-i\delta t). \]  

(6)

The parameter \(\delta\) is the frequency detuning of the carrier wave from the band edge into the gap [19].

For large detunings within the gap, the NLSE model is no longer valid. Instead, one can derive a set of nonlinear coupled mode equations that account for forward and backward propagating wave amplitudes (labelled \(E_+\) and \(E_-\), respectively), and allow for a linear coupling \((\kappa)\) via Bragg scattering, and a nonlinear coupling \((\Gamma)\) via the Kerr effect. When both couplings are weak, the equations take on the form

\[ i\frac{n}{c} \frac{\partial E_+}{\partial t} + i\frac{\partial E_+}{\partial x} + \kappa E_- + \Gamma(|E_+|^2 + 2|E_-|^2)E_+ = 0, \]

\[ i\frac{n}{c} \frac{\partial E_-}{\partial t} - i\frac{\partial E_-}{\partial x} + \kappa E_+ + \Gamma(|E_-|^2 + 2|E_+|^2)E_- = 0, \]  

(7)

where \(\kappa\) is defined in terms of the index modulation, \(\Delta n\), of the PBG material relative to its average index \(\bar{n}\),

\[ \kappa = \frac{n_0}{2n} \cdot k_0, \]  

(8)

and the nonlinear coefficient, \(\Gamma\), is

\[ \Gamma = \frac{6\pi}{\bar{n}^2} \chi^{(3)} k_0. \]  

(9)

In defining these quantities we have introduced the Bragg wavenumber, \(k_0\), associated with the spatial period, \(a\), via \(k_0 = \pi/a\). Christodoulides and Joseph [24] as well as Aceves and Wabnitz [25] have shown that the NLCMEs admit solitary wave solutions which, in principle, are not robust with respect to collision. Furthermore, the NLCMEs include the effects of the dispersion relation of the PBG for a large range of frequencies around the band gap. Therefore, they can be used to describe the interaction of two pulses with different carrier frequencies by making the appropriate ansatz for \(E_+\) and \(E_-\).

It has been shown via a multiple scales analysis that solutions of the NLCME can be related to solutions of the NLSE for those frequency regimes where the NLSE is valid. In essence, the NLCMEs are the more accurate model of the two equations, so that any physical insights gained via the NLSE should be verified by direct simulation of the NLCMEs [35].

3. Non-resonant wave interaction

In general, the interaction between solitary waves is inelastic [36]. However, when the underlying equations of motion are integrable, solitary waves interact elastically [37]. That is, after interaction the waves regain their initial shape and velocity. The only remnant effect of the interaction is a phase shift, which can be translated directly into a wave front shift with respect to the noninteracting case.
Both elastic and inelastic collisions can be realized within the NLSE model. Elastic collisions, generally referred to as resonant interactions, take place when the interacting solitons have very similar carrier frequencies and, consequently, possess similar group velocities and GVDs. This means that the interacting pulses obey the NLSE with the same coefficients; and the resonant interaction processes can be described through the N-soliton solution of the NLSE [38,39]. Depending on the initial conditions, various resonant effects such as soliton trapping and the formation of bound states can be realized [36]. The spectral overlap of the interacting solitons is essential for resonant interaction. If this overlap is zero or negligible, the solitons do not interfere coherently and resonant interaction does not occur.

Inelastic collisions occur in the so-called non-resonant regime [40,41]. In this regime, the interacting solitons have sufficiently different group velocities that they pass through each other quickly, thus avoiding resonant effects. Note, in this regime, the interacting solitons obey different NLSEs with different coefficients. In a PBG material, for example, a non-resonant interaction could involve a stationary gap soliton and a propagating Bragg soliton, as illustrated in Fig. 1. The wave front shifts associated with non-resonant interactions in PBG materials may have potential for applications such as all-optical buffers, logic gates, etc. [16]. In particular, as discussed in the introduction, non-resonant interaction in PBG materials may be used: (i) to launch stationary gap solitons by using Bragg solitons to pull them into the PBG material; (ii) to probe the existence and parameters of stationary gap solitons through the corresponding shift of the colliding Bragg soliton; and (iii) to control and manipulate the position of a stationary gap soliton. However, we emphasize these statements are valid only for the integrable NLSE limit. When the equations of motion that govern the nonlinear dynamics are non-integrable, inelastic effects are present during the interaction processes. Below, we will use inelastic interaction processes within the NLCMEs synonymously with resonant processes. Conversely, elastic processes within the NLCMEs can be reduced to the NLSE model and may be resonant or non-resonant. However, the non-resonant regime remains associated with the NLSE regime only.

3.1. Multiple scales analysis of non-resonant interaction

Here, we use the method of multiple scales to obtain general analytical expressions for the nonlinear phase shift induced by the non-resonant interaction of solitons. We consider two pulses with different carrier frequencies $\omega_{m_i}$ ($i = 1, 2$). These pulses may be in the same band, $n$, at different wave vectors, $k_1$ and $k_2$, or in different bands, $n_1$ and $n_2$, at the same or different wave vectors.

As mentioned, the method of multiple scales has been widely used to treat nonlinear problems in PBG materials, and the reader is referred to those sources for more details on the method. The basic idea is to introduce a small parameter, $\mu$, that defines the smallness of the nonlinear wave amplitude. We then write the electric field, $E(x, t)$, as [19]

$$E(x, t) \equiv \mu \, e(x, t),$$

(10)

where $e(x, t)$ includes all perturbations which arise from weak nonlinear processes:

$$e(x, t) = e_1(x, t) + \mu \, e_2(x, t) + \mu^2 \, e_3(x, t) + \cdots.$$

(11)

Next, we formally replace the space and time variables, $x$ and $t$, with sets of independent spatial and temporal variables $\{x_n \equiv \mu^n x\}$ and $\{t_n \equiv \mu^n t\}$, where $n = 0, 1, 2, \ldots$ [22]. With this replacement, the functions, $e_i$, depend on all $x_n$ and all $t_n$: $e_i(x) \equiv e_i(\{x_n\}, \{t_n\})$.

However, for a perfectly periodic PC, the material parameters are functions of the smallest length scale $x_0$ only, i.e. $e(x) \equiv e(x_0)$ and $\chi^{(3)}(x) \equiv \chi^{(3)}(x_0)$. For the purposes of differentiation, the different spatial and temporal variables are considered to be independent.

In our problem, we use the following ansatz for $e_1$:

$$e_1 = \sum_{i=1}^{2} A_i(\eta_1, \eta_2, \tau) \varphi_{m_i}(x_0) \exp \left[ -i \omega_{m_0} t_0 + i \mu \Omega_{m_i} \right]$$

$$+ c.c.,$$

(12)

where $\varphi_{m_1}(x_0)$ and $\varphi_{m_2}(x_0)$ are the carrier waves (Bloch functions) of the interacting pulses. $A_i(\eta_1, \eta_2, \tau)$ denotes the slowly varying envelope function of the $i$th wave, and $\Omega_{m_i} = \Omega_{m_i}(\eta_1, \eta_2, \tau)$ is the phase shift of the $i$th wave after the interaction. The quantities, $\eta_1$ and $\eta_2$, are related to the Galilean frame of reference travelling with the respective pulses. In the absence of the second pulse, $\eta_1$ would be given by $\eta_1 = \mu (x - v g_1 t)$, where $v g_1$ is the group velocity of pulse one at $\omega_{m_1}$. However, since the pulses interact, we assume that $A_i(\eta_1, \eta_2, \tau)$ and $\Omega_{m_i}(\eta_1, \eta_2, \tau)$ depend on the slow time variable,

$$\tau = \mu^2 t.$$  

(13)
This means that the Galilean frames of the respective pulses will also be slightly affected, at least during their interaction. Consequently, we allow \( \eta_1 \) and \( \eta_2 \) to vary during the interaction:

\[
\eta_1 = \mu (x - v_{g1} t - \mu \psi_{m_1}(\eta_1, \eta_2, \tau));
\]

\[
\eta_2 = \mu (x - v_{g2} t - \mu \psi_{m_2}(\eta_1, \eta_2, \tau)).
\]

To generate a NLSE description, it is sufficient to restrict the analysis to \( \eta_i (i = 1, 2) \) and \( \tau \) only. Clearly, the carrier wave phase and wave front shifts, \( \Omega_m \) and \( \psi_{m_i} \), are not independent of each other. In the following, we determine \( \Omega_m \) from the physical parameters of the system, and subsequently determine \( \psi_{m_i} \).

From Eqs. (13)–(15), the first order derivatives of an arbitrary function \( Y(x_0, t_0, \eta_1, \eta_2, \tau) \) are

\[
\frac{dY}{dt} = \frac{\partial Y}{\partial t_0} + \mu \left[ -v_{g1} \frac{\partial Y}{\partial \eta_1} - v_{g2} \frac{\partial Y}{\partial \eta_2} \right] + \mu^2 \frac{\partial^2 Y}{\partial \tau^2};
\]

\[
\frac{dY}{dx} = \frac{\partial Y}{\partial x_0} + \mu \left[ \frac{\partial Y}{\partial \eta_1} + \frac{\partial Y}{\partial \eta_2} \right];
\]

and the second order derivatives are:

\[
\frac{d^2 Y}{dt^2} = \frac{\partial^2 Y}{\partial t_0^2} + \mu^2 \left[ v_{g1}^2 \frac{\partial^2 Y}{\partial \eta_1^2} + 2v_{g1}v_{g2} \frac{\partial^2 Y}{\partial \eta_1 \partial \eta_2} + v_{g2}^2 \frac{\partial^2 Y}{\partial \eta_2^2} \right];
\]

\[
\frac{d^2 Y}{dx^2} = \frac{\partial^2 Y}{\partial x_0^2} + \mu^2 \left[ \frac{\partial^2 Y}{\partial \eta_1^2} + 2 \frac{\partial^2 Y}{\partial \eta_1 \partial \eta_2} + \frac{\partial^2 Y}{\partial \eta_2^2} \right].
\]

Terms of order \( \sim \mu^3 \) and higher are neglected in the leading order of nonlinear effects considered here.

We now substitute Eqs. (10)–(12) into the wave Eq. (1) and collect terms with the same power of \( \mu \). This determines the equations of motion for the corresponding time and spatial scales.

### 3.1.1. The first order multi-scale analysis

In the lowest (first) order in \( \mu \), we find:

\[
\left[ -c^2 \frac{\partial^2}{\partial x_0^2} + \varepsilon(x_0) \frac{\partial^2}{\partial t_0^2} \right] e_1 = 0.
\]

Since this is just the wave equation for the PBG material in the absence of nonlinearity, we find immediately that \( \varphi_{m_1}(x_0) \) and \( \varphi_{m_2}(x_0) \) represent Bloch functions with corresponding frequencies, \( \omega_{m_1} \) and \( \omega_{m_2} \), respectively.

### 3.1.2. The second order multi-scale analysis

To second order (terms of order \( \mu^2 \)), we find:

\[
\left[ -c^2 \frac{\partial^2}{\partial x_0^2} + \varepsilon(x_0) \frac{\partial^2}{\partial t_0^2} \right] e_2 = \mathcal{R}^{(2)},
\]

where \( \mathcal{R}^{(2)} \) is given by

\[
\mathcal{R}^{(2)} = 2 \left[ c^2 \left( \frac{\partial A_1}{\partial \eta_1} + \frac{\partial A_1}{\partial \eta_2} \right) \frac{\partial}{\partial x_0} \varphi_{m_1}(x_0) \right.

- 2i \omega_m \varepsilon(x_0) \left( v_{g1} \frac{\partial A_1}{\partial \eta_1} + v_{g2} \frac{\partial A_1}{\partial \eta_2} \right) \varphi_{m_1}(x_0) \exp(-i \omega_m t_0)

+ \left[ c^2 \left( \frac{\partial A_2}{\partial \eta_1} + \frac{\partial A_2}{\partial \eta_2} \right) \frac{\partial}{\partial x_0} \varphi_{m_2}(x_0) \right.

- 2i \omega_m \varepsilon(x_0) \left( v_{g1} \frac{\partial A_2}{\partial \eta_1} + v_{g2} \frac{\partial A_2}{\partial \eta_2} \right) \varphi_{m_2}(x_0) \exp(-i \omega_m t_0) + c.c.

\]

We analyze Eqs. (21) and (22) using the following Ansatz for \( e_2 \):

\[
e_2 = \sum_{l_1=1}^{\infty} B_{l_1}(\eta_1, \eta_2, \tau) \varphi_{l_1}(x_0) \exp(-i \omega_{m_1} t_0)

+ \sum_{l_2=1}^{\infty} B_{l_2}(\eta_1, \eta_2, \tau) \varphi_{l_2}(x_0) \exp(-i \omega_{m_2} t_0) + c.c.
\]

Here the sum over \( l_1 \) runs over all band indices at the same (fixed) wave vector \( k_1 \) associated with the carrier wave \( m_1 \equiv (n_1, k_1) \). Similarly, the sum over \( l_2 \) extends over all bands at the wave vector \( k_2 \) associated with the carrier wave \( m_2 \equiv (n_2, k_2) \).

Substituting (23) into Eqs. (21) and (22), we find

\[
\sum_{l=1}^{\infty} (\omega_{l_1}^2 - \omega_{m_1}^2) \varepsilon(x_0) \varphi_{l_1}(x_0) B_{l_1}

= 2i \left[ \varepsilon \left( \frac{\partial A_1}{\partial \eta_1} + \frac{\partial A_1}{\partial \eta_2} \right) \left( \hat{p} \varphi_{m_1}(x_0) \right) \right.

- \omega_m \varepsilon(x_0) \left( v_{g1} \frac{\partial A_1}{\partial \eta_1} + v_{g2} \frac{\partial A_1}{\partial \eta_2} \right) \varphi_{m_1}(x_0) \right],
\]

where

\[
\hat{p} = -i c \frac{\partial}{\partial x_0}.
\]
Projecting Eq. (24) onto the carrier waves, \( \varphi_m(x_0) \), results in:

\[
(v_{g1} - v_{g2}) \frac{\partial A_1}{\partial \eta_2} = 0; \tag{26}
\]

\[
(v_{g1} - v_{g2}) \frac{\partial A_2}{\partial \eta_1} = 0, \tag{27}
\]

where we have made use of the orthonormality of the Bloch functions (see Eq. (3)). Eqs. (26) and (27) determine the quantitative condition for the realization of the non-resonant interaction regime:

\[
\left| \frac{v_{g1} - v_{g2}}{v_{g1}} \right| \gg \mu \quad (i = 1, 2). \tag{28}
\]

When the group velocity of one of the interacting waves, say, the first wave, is zero \( v_{g1} = 0 \), Eq. (28) must be replaced by \( n v_{g2}/c \gg \mu \), where \( n \) is the average background index of refraction. These conditions ensure that the interaction time of the two pulses is sufficiently short that no resonant energy transfer can take place.

We note that, according to Eqs. (26)–(28), in the non-resonant interaction regime, the nonlinear wave envelopes, \( A_1 \) and \( A_2 \), are decoupled, i.e.,

\[
\frac{\partial A_1}{\partial \eta_2} = \frac{\partial A_2}{\partial \eta_1} = 0, \tag{29}
\]

so that \( A_i(\eta_1, \eta_2, \tau) = A_i(\eta_1, \tau) \). In addition, projecting Eq. (24) onto all Bloch functions \( \varphi_m(x_0) \) at the same wave vector as band \( m_i \) \( (i \neq m_i) \), we obtain expressions for the secondary envelope functions \( B_i \),

\[
B_i = B_i(\eta_i, \tau) = \left\langle \psi_i | \psi_{m_i} \right\rangle \frac{\partial A_i}{\partial \eta_i}. \tag{30}
\]

Consequently, the secondary envelope functions, \( B_i \), associated with the interacting waves are decoupled, too. These secondary envelope functions ultimately provide the group velocity dispersion for the primary amplitudes, \( A_m \) (see [19]).

### 3.1.3. The third order multi-scale analysis

Finally, to third order in \( \mu \), we find

\[
\left[ -c^2 \frac{\partial^2}{\partial \xi^2} + \varepsilon(x_0) \frac{\partial^2}{\partial t^2} \right] e_3
= 4\pi \omega_m^2 \chi^{(3)}(x_0) \mathcal{R}_1^{(3)} + \mathcal{R}_2^{(3)} + \mathcal{R}_3^{(3)}, \tag{31}
\]

where only \( \mathcal{R}_1^{(3)} \) originates from the nonlinearity,

\[
\mathcal{R}_1^{(3)} = 3 \varphi_m |\varphi_m| A_1^2 A_1 + 6 \varphi_m^2 \varphi_m |A_2| A_1 \exp(-i\omega_m t_0)
+ 3 \varphi_m^2 \varphi_m |A_2| A_2 \exp(-i\omega_m t_0) + c.c. \tag{32}
\]

In deriving \( \mathcal{R}_1^{(3)} \) we have applied the rotating-wave approximation.

The expressions for \( \mathcal{R}_2^{(3)} \) and \( \mathcal{R}_3^{(3)} \) are

\[
\mathcal{R}_2^{(3)} = \sum_{i=1}^2 2i\omega_m \sum_{l=1}^\infty \left[ \frac{c}{\alpha_m} \left( \hat{p} \varphi_i(x_0) - v_{g1} \varphi_i(x_0) \right) \right]
\times \frac{\partial B_i}{\partial \eta_i} \exp(-i\omega_m t_0) + c.c., \tag{33}
\]

and

\[
\mathcal{R}_3^{(3)} = \sum_{i=1}^2 \left[ c^2 \varphi_m(x_0) \frac{\partial^2}{\partial \eta_i^2} A_i + 2i c^2 \left( \frac{\partial \Omega_m}{\partial \eta_1} + \frac{\partial \Omega_m}{\partial \eta_2} \right) \right]
\times A_i \frac{\partial}{\partial x_0} \varphi_m(x_0) - \varepsilon(x_0) \varphi_m(x_0) \frac{\partial^2}{\partial \eta_i^2} A_i
+ 2i\omega_m \varepsilon(x_0) \varphi_m(x_0) \frac{\partial}{\partial \eta_i} A_i
+ 2\omega_m \left( v_{g1} \frac{\partial \Omega_m}{\partial \eta_1} + v_{g2} \frac{\partial \Omega_m}{\partial \eta_2} \right) \varphi_m(x_0) A_i \right]
\times \exp(-i\omega_m t_0) + c.c. \tag{34}
\]

The appropriate Ansatz for \( e_3 \) is

\[
e_3 = \sum_{l=1}^\infty C_{l1}(\eta_1, \eta_2, \tau) \varphi_{l1}(x_0) \exp(-i\omega_m t_0)
+ \sum_{l=2}^\infty C_{l2}(\eta_1, \eta_2, \tau) \varphi_{l2}(x_0) \exp(-i\omega_m t_0) + c.c., \tag{35}
\]

where the sums obey the same rules as those in Eq. (23).

Substituting Eq. (35) into Eq. (31) and projecting the resulting expression onto \( \varphi_{m1}(x_0) \), we find that, as expected, the envelope functions \( A_i \) of the interacting pulses each obey the NLSE with corresponding coefficients

\[
\frac{i}{2} \frac{\partial A_i}{\partial t} + \frac{1}{2} \alpha_m \frac{\partial^2 A_i}{\partial \eta_i^2} + \chi^{(3)}(x_0) |A_i|^2 A_i = 0. \tag{36}
\]

More importantly, we find that the nonlinear phase shift \( \Omega_m \) of the first wave depends only on the difference in group velocities, the effective cross phase
modulation $\Delta_{\text{eff}}^{(3)}$ between the pulses and the amplitude of the second wave

$$\frac{\partial}{\partial \eta_2} \Omega_{m_1} = \frac{\Delta_{\text{eff}}^{(3)}}{v_{g_1} - v_{g_2}} |A_2|^2. \quad (37)$$

The cross-phase modulation constant $\Delta_{\text{eff}}$ is defined as (cf. Eq. (5))

$$\Delta_{\text{eff}}^{(3)} = 12\pi\omega_{m_1} \int \chi^{(3)}(x_0)|\varphi_{m_1}(x_0)|^2|\varphi_{m_2}(x_0)|^2 \, dx_0. \quad (38)$$

We note that, as a consequence of Eq. (37), $\Omega_{m_1}$ depends only on $\eta_2$ and is independent of $\eta_1$. The phase shift of the second wave is given by a completely analogous expression with the indices 1 and 2 interchanged.

Eq. (38) suggests that the interaction between nonlinear pulses in the non-resonant regime is determined by the overlap of the carrier Bloch functions of the pulses. In particular, it is possible that the cross-phase modulation between the pulses is enhanced, due to the large overlap of the Bloch functions, $\varphi_{m_1}(x_0)$ and $\varphi_{m_2}(x_0)$, and the Kerr nonlinearity $\chi^{(3)}(x_0)$. On the other hand, when $\chi^{(3)}(x_0)$ changes sign within the unit cell, it may happen that for certain pairs of frequencies the overlap of the corresponding Bloch functions is zero. In such situations non-resonant interaction effects would be entirely absent.

### 3.2. The wave front shift

We now determine the wave front shift $\psi_{m_1}$ associated with the nonlinear phase shift $\Omega_{m_1}$. We consider the group velocity of the first wave during the interaction, which can be re-written as

$$\frac{d(\omega_{m_1} + \Delta\omega_{m_1})}{d(k_1 + \Delta k_1)} = \left(1 - \frac{d\Delta k_1}{dk_1}\right) \left(\frac{d\omega_{m_1}}{dk_1} + \frac{d\Delta\omega_{m_1}}{dk_1}\right) \approx \frac{d}{dk_1} \omega_{m_1}$$

$$-\frac{d}{dk_1} \Delta k_1 \frac{d}{dk_1} \omega_{m_1} + \frac{d}{dk_1} \Delta\omega_{m_1}. \quad (39)$$

Here, we have introduced the nonlinear wave vector shift, $\Delta k_1$, and the nonlinear frequency shift, $\Delta\omega_{m_1}$, experienced by the first wave via interaction with the second:

$$\Delta k_1 = \mu \frac{\partial}{\partial x} \Omega_{m_1} = \mu^2 \frac{\partial}{\partial \eta_2} \Omega_{m_1} \quad (40)$$

$$\Delta\omega_{m_1} = -\mu \frac{\partial}{\partial t} \Omega_{m_1} = \mu^2 v_{g_2} \frac{\partial}{\partial \eta_2} \Omega_{m_1} \equiv v_{g_2} \Delta k_1. \quad (41)$$

In deriving these expressions, we used Eqs. (16) and (17).

Thus, the nonlinear group velocity shift $\Delta v_{g_1}$ of the first pulse during the interaction process becomes

$$\Delta v_{g_1} \equiv \frac{d(\omega_{m_1} + \Delta\omega_{m_1})}{d(k_1 + \Delta k_1)} - \frac{d}{dk_1} \omega_{m_1}$$

$$= \frac{d}{dk_1} \left[\Delta\omega_{m_1} - \left(\frac{d}{dk_1} \omega_{m_1}\right) \Delta k_1\right]$$

$$= -\mu^2 (v_{g_1} - v_{g_2}) \frac{d}{dk_1} \left(\frac{\partial}{\partial \eta_2} \Omega_{m_1}\right). \quad (42)$$

These considerations suggest a clear physical interpretation of the non-resonant wave interaction. The local distortion of the photonic band structure induced by one of the pulses appears to the other pulse as a shift in the local group velocity. This is a reciprocal effect and must be summed up over the duration of the interaction process to produce the anticipated wave fronts shifts of the interacting pulses relative to the non-interacting case.

To carry out this summation, we derive from Eq. (14) that

$$\left.\frac{d\eta_1}{dr}\right|_{\eta_1=\text{const}} = \left.\frac{dx}{dr}\right|_{\eta_1=\text{const}} - v_{g_1} - \mu \frac{\partial}{\partial t} \psi_{m_1} = 0. \quad (43)$$

Thus, we find an alternative expression for the nonlinear group velocity shift, $\Delta v_{g_1}$:

$$\Delta v_{g_1} = \left.\frac{dx}{dr}\right|_{\eta_1=\text{const}} - v_{g_1} = \mu \frac{\partial}{\partial t} \psi_{m_1}, \quad (44)$$

which directly implies the relation

$$\Delta v_{g_1} = \mu \frac{\partial \eta_2}{\partial r}\left|_{\eta_1=\text{const}} \right. \frac{\partial}{\partial \eta_2} \psi_{m_1}$$

$$= \mu^2 (v_{g_1} - v_{g_2}) \frac{\partial}{\partial \eta_2} \psi_{m_1}. \quad (45)$$

Based on Eqs. (42) and (45), we may now identify

$$\frac{\partial}{\partial \eta_2} \psi_{m_1} = -\frac{\partial}{\partial k_1} \left(\frac{\partial}{\partial \eta_2} \Omega_{m_1}\right). \quad (46)$$

Eqs. (37), (38), and (46) give a complete description of non-resonant interaction in the system. For instance, from Eq. (46) we can calculate the total wave front shift, $\Delta l_1$, of the first wave envelope $A_1$:
\[ \Delta l_1 = \int_{-\infty}^{+\infty} \frac{\partial}{\partial \eta_2} \psi_{m_1} \, d\eta_2 \]
\[ = \frac{\partial}{\partial k_1} \left[ \frac{\Delta_{\text{eff}1}^{(3)}}{(v_{g2} - v_{g1})} \right] \int_{-\infty}^{+\infty} |A_2|^2 \, d\eta_2. \tag{47} \]

The second wave envelope, \( A_2 \), experiences an analogous shift, with indices changing from 1 to 2 and vice-versa. The implications of Eq. (47) are: (i) we can control and modify the position of a nonlinear pulse (pulse one) by controlling the duration and intensity of a collision partner (pulse two) and (ii) the collision partner (pulse two) acquires a wave front shift that provides information about the parameters of pulse one. Note, Eq. (47) is valid for pulses well-described by the NLSE, i.e., for collisions of Bragg solitons with Bragg or gap soliton as well as for the collision of gap solitons with gap solitons. In addition, Eq. (47) also describes the interaction of extended waves with localized pulses. However, in what follows we concentrate on collisions between Bragg solitons and stationary gap solitons.

### 3.3. An example: fiber Bragg gratings

We now apply the formulae derived in the previous section to a system with the physical parameters of a fiber Bragg grating, which consists of a periodic modulation of the index of refraction along the core of an optical fiber [42]. The refractive index modulation is usually weak (\( \Delta n/n \leq 10^{-3} \)) and the nonlinear processes are well described by the nonlinear coupled mode equations [28,26].

We consider the wave front shift, \( \Delta l_1 \), of a stationary \((v_{g1} \equiv 0)\) gap soliton induced by a collision with a propagating Bragg soliton \((v_{g2} \neq 0)\). We first note that for a 1D PC, the properly normalized (see Eq. 3) Bloch functions in the upper band of the dispersion relation can be written as:

\[ \varphi_{m_1}(x_0) = \frac{\exp(i k x_0)}{\sqrt{2 a n}} \left[ \sqrt{1 + \bar{v}_g} + \sqrt{1 - \bar{v}_g} \exp(-2i k_0 x_0) \right]. \tag{48} \]

The scaled group velocity, \( \bar{v}_{g1} = \bar{n} v_{g1}/c \), is defined with respect to the group velocity for frequencies well away from a Bragg resonance. Furthermore, we assume that the nonlinear Kerr coefficient is constant along the fiber \((\chi^{(3)}(x_0) = \chi^{(3)}_0)\). We then find that the cross phase modulation constant (38) is:

\[ \Delta_{\text{eff}1}^{(3)} = \frac{12 \pi \omega_m \chi^{(3)}_0}{a n^2} \left( 1 + \frac{1}{2} \sqrt{(1 - \bar{v}_{g1}^2)(1 - \bar{v}_{g2}^2)} \right). \tag{49} \]

The intensity profile of the moving Bragg soliton with the characteristic width, \( A_2 \), is given by (see, for instance, [38]):

\[ |A_2|^2 = \frac{\omega_{m2}^2}{\chi_{\text{eff}2}^{(3)} A_2^2} \frac{1}{\cosh^2(\eta_2/A_2)}. \tag{50} \]

And using (5) and (48), the effective nonlinearity (50) is:

\[ \chi_{\text{eff}2}^{(3)} = \frac{3 \pi \omega_{m2}}{a n^2} \chi^{(3)}_0 (3 - \bar{v}_{g2}^2). \tag{51} \]

Substituting Eqs. (49)–(51) into Eq. (47) we arrive at the wave front shift, \( \Delta l_1 \), of the stationary gap soliton shift after the collision:

\[ \Delta l_1 = \frac{8}{\kappa^2 A_2} \frac{(1 - \bar{v}_{g2}^2)^2}{\bar{v}_{g2}^2(3 - \bar{v}_{g2}^2)} \left[ 1 + \frac{1}{2} \sqrt{1 - \bar{v}_{g2}^2} \right]. \tag{52} \]

Analogously, the shift of the Bragg soliton, \( \Delta l_2 \), is

\[ \Delta l_2 = \frac{8}{3 \kappa^2 A_1} \left[ \frac{1 - 2 \bar{v}_{g2}^2}{\bar{v}_{g2}^2} \left( 1 + \frac{1}{2} \sqrt{1 - \bar{v}_{g2}^2} \right) + \frac{1}{2} \sqrt{1 - \bar{v}_{g2}^2} \right]. \tag{53} \]

In deriving (53) we have used the analytical expression in [28] for the dispersion relation to calculate the GVD of the two pulses.

From Eq. (52) it is apparent that to obtain a large displacement of the stationary gap soliton, the values of \( A_2 \) and \( \bar{v}_{g2} \) should be as small as possible. This is expected, because the smaller the value of \( A_2 \), the larger the intensity of the moving soliton; similarly, for smaller values of \( \bar{v}_{g2} \), solitons interact for a longer time. The soliton shifts are inversely proportional to the square of the grating strength, \( \kappa \), because for smaller values of \( \kappa \), the group velocity dispersion at the band edge is larger [28], and consequently, the interacting solitons have more energy. Finally, once the grating strength, \( \kappa \), and the scaled velocity, \( \bar{v}_{g2} \), of the Bragg soliton are known, the Bragg soliton shift, \( \Delta l_2 \), allows a direct determination of the width, \( A_1 \), of the stationary gap soliton.

In the following section, we subject Eqs. (52) and (53) to a detailed comparison with direct numerical simulations of the NLCMEs (see also [29]).
4. Numerical simulations and comparison with analytical results

In this section we use the NLCMEs to simulate pulse collision for a wide range of pulse velocities and frequency detunings. The simulated pulses are the solitary wave solutions to the NCLMEs proposed by Aceves and Wabnitz [25]. The NLCMEs are non-integrable, so these solitary waves do not necessarily collide elastically. Nevertheless, we do find parameter regimes in which the collisions are non-resonant (and are thus well-described by the analytical predictions of the previous section). In other regimes the collisions are resonant, and exhibit complicated behaviour. The identifying mark of the resonant regime is that the outcome of the collisions is sensitive to the phase difference between the pulses.

We first estimate time and length scales for realistic material parameters. A typical fiber Bragg grating exhibits an average index of refraction, \( \bar{n} = 1.45 \) and an nonlinear refractive index, \( n_2 = \chi_0^{(3)}/(2\bar{n}) = 2.3 \times 10^{-20} \text{ m}^2/\text{W} \). Typical experiments [43] for Bragg and gap solitons in fiber gratings have used a Bragg wavelength, \( \lambda_0 = 1053 \text{ nm} \), and an index modulation on the order of \( \Delta n = 3 \times 10^{-4} \). These parameters give time and length scales \( T = \bar{n}/c\kappa = 5.4 \times 10^{-12} \text{ s} \) and \( X = 1/\kappa = 1.11 \text{ mm} \), respectively. We also introduce the dimensionless soliton width, \( L_i \), and the soliton shift, \( \delta l_i \), through \( L_i = XL_i \) and \( \Delta l_i = X\delta l_i \).

4.1. Resonant versus non-resonant interaction

To map out the regimes where resonant and non-resonant interactions take place, we simulated collisions between two Aceves and Wabnitz solitary waves with identical detunings and opposite velocities, but with a varying relative phase, \( \Delta \Phi_0 \). The results of the simulations are summarized in Fig. 3, using \( \delta \), the frequency detuning from the band gap edge (see Eq. (6)), and \( v \), the group velocity of the wave scaled to the velocity of light far from a band gap, as control parameters. The \( (\delta, v) \)-parameter plane is separated into two distinct regions. For sufficiently wide and fast solitons, the phase difference between the pulses has no effects on the result at all and the interaction process is of a non-resonant nature. We note that even in the non-resonant regime, the intensity distribution during the interaction does look slightly different for different relative phase differences, \( \Delta \Phi_0 \). Nevertheless, Fig. 4 demonstrates that the shapes of the solitons before and after the collision are identical. In fact, the solitons interact elastically and regain their detuning and velocity after the interaction, thus justifying our nomenclature of the non-resonant interaction regime. The only remnant effect of the collision is a shift of the wave front relative to the non-interacting case (see Fig. 4).

By contrast, the interaction of relatively narrow and/or slow solitons strongly depends on the initial phase difference \( \Delta \Phi_0 \). This is demonstrated in Fig. 5, where we display several outcomes for counter-propagating gap solitons (\( \delta = 0.3\pi \) and \( v = \pm 0.5 \) ) for different values of the relative phase \( \Delta \Phi_0 \). Obviously, besides the...
soliton position, the shapes of the pulses after interaction are significantly different for different relative phases, $\Delta \Phi_0$. The nonlinear wave interaction in this case is clearly resonant. In general, we find that out-of-phase solitons ($\Delta \Phi_0 = \pi$) interact repulsively. The centers of the solitons never “touch” and as long as the solitons are wide and exhibit a sufficiently small detuning, the interaction remains elastic. If the detuning is too large, the solitons shed radiation and the collision ceases to be elastic. The actual value of the velocity is only of secondary interest in these cases. In some extreme cases the solitons are completely destroyed. In

For solitons that are in-phase ($\Delta \Phi_0 = 0$), the interaction dynamics exhibits a much more complicated behavior. In general, in-phase solitons interact attractively, which means that during the interaction all the energy is concentrated into a single sharp intensity peak. The outcome of the collision depends strongly on both the detuning and (somewhat less strong) on the velocity. In the corresponding “phase diagram” Fig. 6b, we can identify at least four different interaction regimes.

For low values of the detuning, we are in the NLSE limit where collisions are elastic. But within a region of the detuning parameter between $0.18 \pi$ and $0.3 \pi$ and for velocities below $v = 0.2$, fusion of the colliding solitons into one stationary pulse takes place. The transition from the elastic scattering regime to the fusion regime is rather abrupt. Very close to the border between these regimes, we observe processes where the solitons interact multiple times before finally separating. These processes are accompanied by a conspicuous spontaneous symmetry breaking, which means the resulting solitons are no longer symmetric with respect to the systems’ center of mass. Most probably the reason for this is numerical noise, which triggers an instability of the high-intensity peaks that form during the interaction process. This instability eventually leads to a breaking of symmetry. In [30], Mak et al. suggest utilizing soliton fusion for creating stationary light pulses. However, our detailed numerical calculations indicate that in order to realize this

Fig. 6a, we display the boundary between elastic and inelastic regime for the repulsive case.

Fig. 5. Examples of resonant interaction processes. We show the outcome of a collision between two gap solitons with $\delta = 0.3\pi$ and $v = \pm 0.5$ at $t = 80$ for different values of relative phase, $\Delta \Phi_0$. In contrast to the non-resonant case (Fig. 4), the shapes of the pulses after interaction vary dramatically with $\Delta \Phi_0$. Also, the velocities of the emerging solitons strongly deviate from the launching velocity so that at $t = 80$, the positions of the pulses vary.

Fig. 6. (a) “Phase diagram” for the repulsive interaction of out-of-phase ($\Delta \Phi_0 = \pi$) pulses. The interaction changes from elastic to inelastic for sufficiently high detuning. (b) “Phase diagram” for the attractive interaction of in-phase $\Delta \Phi_0 = 0$ solitons. At least four parameter regimes can be identified. Soliton fusion occurs in the parameter region marked by “F” which is separated from other region by abrupt boundaries. In contrast, the dashed lines indicate “continuous” (or soft) transitions, between parameters regions of elastic, quasi-elastic, and strongly inelastic processes.
soliton fusion, the relative phase difference $\Delta \Phi_0 = 0$ between the pulses needs to be controlled with a relative precision of better than $10^{-4}$. Otherwise, the fusion to a stationary soliton does not take place and the interacting pulses eventually separate from each other after a highly complex interaction process. Therefore, the experimental realization of the soliton fusion process may prove to be challenging. As alluded to above, for small detunings we find a regime of elastic interactions where the NLSE is valid. However, for intermediate detunings, the solitons interact quasi-elastically, meaning their final velocity and energy after the collision change relative to their launching values, but their radiation losses are small (typically below 1% of the total energy). Very close to the borders with the fusion region, the solitons slow down after colliding with each other. Everywhere else within the quasi-elastic regime, the final velocities are higher than the initial velocities. This difference in velocities decreases with increasing initial velocities. Finally, for large detunings, we enter an interaction regime where the amount of radiation losses becomes so large that during the interaction the solitons are completely destroyed. We refer to this strongly inelastic regime as the region of strong deformation. Except for the transition to the fusion regime, transitions between any of the regimes are “continuous” and in both the quasi-elastic and the strong deformation region all processes lead to radiation losses.

4.2. Giant soliton shifts

As alluded to in the previous sections, in the non-resonant limit the collision between solitons is elastic and their relative phase difference is of no consequence. The only remnant effects of such interaction processes are phase shifts associated with the soliton carrier waves. Since the phase shift of a wave can be directly translated into a corresponding wave front shift, after the interaction process the center of each soliton will be shifted relative to the noninteracting case. We have given the analytical description of these processes within the NLSE model in the previous section (see also [29]). Here, we compare the analytical results of Eqs. (52) and (53) via numerical simulations of collisions between a stationary gap and moving Bragg solitons using the NLCMEs. The results of these simulations for widths $L_1 = 5$, and $L_2$ of the stationary gap soliton and widths $L_2 = 5, 10, 20, 40$ of the Bragg soliton are displayed in Figs. 7 and 8. We note that fixing the width of the soliton uniquely determines all soliton parameters for any given velocity [35]. In all the cases described above, the corresponding detunings are well below $0.15\pi$, which means that the NLSE is valid. The shifts, $\delta l_1$, of the stationary gap soliton are plotted in the left sub-figures; whereas, the corresponding shifts, $\delta l_2$, of the Bragg soliton are presented in the right sub-figures. In each case, we fix the width, $L_1$, of the stationary gap soliton and vary the width, $L_2$, of the Bragg moving soliton. For comparison, we also plot the analytic predictions (given by solid lines) based on Eqs. (52) and (53). From Eqs. (52) and (53), it is obvious that the analytic expression for the shift of one soliton depends uniquely on the width of its collision partner. As a result, we have that because the width of the stationary gap soliton is fixed for each case, in Figs. 7 and 8, the analytic predictions on the left panels are always different; whereas, there is only one analytical curve on the corresponding right panels.

Figs. 7 and 8 suggest that the wider the solitons, the better agreement between the NLCMEs and the NLSE predictions. This is expected, since it is precisely within
this limit that the NLCMEs reduce to the NLSE. If both solitons are very narrow, then both wave front shifts start to oscillate as a function of the Bragg soliton’s velocity. This behavior signals that resonant effects are starting to become increasingly important and/or the collisions are no longer elastic. A typical example of non-resonant collisions between gap and Bragg solitons that lead to giant wave front shifts is depicted in the left panel of Fig. 9. A typical inelastic interaction process, as displayed in the right panel of Fig. 9, allows the initially stationary gap soliton becomes mobile after the collision.

For the typical experimental values given above, the soliton shifts are of the order of millimeters or even centimeters. This is several orders of magnitude larger than the corresponding shifts found in ordinary fibers (without a grating). This observation makes nonlinear PBG materials particularly interesting for certain applications of these effects.

5. Conclusions

In conclusion, we have studied, both analytically and numerically, nonlinear wave interaction processes in one-dimensional photonic band gap materials. The non-resonant interaction of nonlinear pulses may lead to efficient mechanisms for dynamically controlling optical waves. In particular, for realistic systems the analytical formulae predict that the wave front shift experienced by these pulses is of the order of millimeters or even centimeters. Such pronounced effects should be easily observable in the laboratory. Moreover, the non-resonant interaction processes can be utilized to control the position of stationary gap solitons and to probe their very existence as well as their physical parameters. Furthermore, the direction of this research suggests that non-resonant interaction might be useful for launching stationary gap solitons.

Numerical simulations confirm the analytical predictions for the regime of non-resonant interactions.
In addition, these simulations have allowed us to investigate the inelastic wave interaction regime, for which analytical results are unavailable. Our comprehensive numerical studies demonstrate that, quite generally, nonlinear wave interaction processes are very sensitive to the relative phase of colliding pulses.

Finally, we would like to note that both the NLCMEs and, in particular, the NLSE govern the nonlinear dynamics of a large variety of physical systems, including nonlinear optical and magnetic systems, Bose–Einstein condensates, etc. This suggests that the results presented in this paper are directly applicable to other periodic structures of current interest. Most notably, Bragg and gap solitons have been discussed [44,45] and very recently observed [46] in Bose–Einstein condensates in optical lattices.

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